TRANSIENT LOW REYNOLDS NUMBER FLOW DEVELOPMENT AROUND A MOVING SPHERE

A THESIS

Presented to

The Faculty of the Division

of Graduate Studies

By

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In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Mechanical Engineering

Georgia Institute of Technology

June, 1977

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AROUND A MOVING SPHERE

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ACKNOWLEDGMENTS

I wish to gratefully acknowledge the patience, guidance and sacrifice of my thesis advisor, Dr. Prateen V. Desai. My enthusiasm and interest in the problem of fluid transport was cultivated through Dr. Desai's instruction.

My indebtedness to Dr. Melvin R. Corley and Dr. R. Carlson for their creative considerations and recommendations while serving on my reading committee cannot be repaid.

I would also like to express my appreciation to Dr. S. P. Kezios, Director, School of Mechanical Engineering, for making available my means of financial aid through the graduate teaching assistantship program and the DuPont Scholarship.

Finally, I wish to express my fondest appreciation to my wife, Barbara, and both our parents, for the invaluable financial and spiritual support with which they sustained me.

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NOMENCLATURE

(r,θ,ϕ)	basic spherical coordinate frame
δ	surface tension acting between the two interfacial fluids
ρ	density of the fluid
μ	absolute viscosity of the fluid
ν	kinematic viscosity of the fluid
C_v	specific heat of the fluid at constant volume
k	thermal conductivity of the fluid
β	thermal coefficient of expansion
λ	thermal coefficient of viscosity
γ	thermal coefficient of thermal conductivity
ε	thermal coefficient of specific heat
χ	thermal coefficient of surface tension
t	elapsed time
z	(=ln r) transformed radial axis frame
$\vec{V}$	velocity vector
V_i	i^{th} component of velocity vector
τ_{ij}	ij^{th} component of stress tensor (see Appendix A)
q_i	i^{th} component of energy flux vector (see Appendix B)
Φ_v	dissipation function
p	thermodynamic pressure
ψ	stream function field

ζ	vorticity function field
T	temperature field
g	gravitational acceleration
g_i	i^{th} component of the gravitational vector
U	freestream velocity relative to the spherical element
a	resolution of computational grid in z-direction
b	resolution of computational grid in θ-direction
c	resolution of finite difference equations in time
r_o	spherical element radius at time t (non-constant)
F	$(= \frac{\zeta}{r \sin\theta})$ deformed vorticity field
G	$(= \zeta r \sin\theta)$ deformed vorticity field
N_{Re}	$(= \frac{2Ur_o}{\nu})$ Reynolds number
N_{Pr}	$(= \frac{k}{\mu c_p})$ Prandtl number
c_p	specific heat of the fluid at constant pressure

<u>Subscripts</u>

i	index of z-direction (stretched spherical coordinates)
j	index of θ-direction
r	radial component
θ	tangential component

<u>Superscripts</u>

+	dimensionless value or property
*	internal fluid value or property
'	value of function or field at the next time step

Special Operators

$\vec{\nabla}$	gradient, differential vector operator, $\vec{\nabla} = [\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, 0]$
∇^2	Laplacian operator in spherical coordinates, $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial}{\partial r}) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta \frac{\partial}{\partial \theta})$
$\frac{D}{Dt}$	substantial time derivative, $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{V}\cdot\vec{\nabla})$
E^2	basic spherical coordinate operator, $E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial \theta}(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta})$
C^2	basic spherical coordinate operator, $C^2 = (\frac{2}{r}\frac{\partial^2}{\partial r \partial \theta} - \frac{\cot\theta}{r}\frac{\partial}{\partial r} - \frac{3}{r^2}\frac{\partial}{\partial \theta})$
$(e^{2z}E^2)$	basic external spherical coordinate operator in transformed coordinates $(e^{2z}E^2) = (\frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial z} + \frac{\partial^2}{\partial \theta^2} - \cot\theta \frac{\partial}{\partial \theta})$
$(e^{2z}C^2)$	basic external spherical coordinate operator in transformed coordinates $(e^{2z}C^2) = (2\frac{\partial^2}{\partial z \partial \theta} - \cot\theta \frac{\partial}{\partial z} - 3\frac{\partial}{\partial \theta})$
$(e^{-2z}E^2)$	basic internal spherical coordinate operator in transformed coordinates $(e^{-2z}E^2) = (\frac{\partial^2}{\partial z^2} + \frac{\partial}{\partial z} + \frac{\partial^2}{\partial \theta^2} - \cot\theta \frac{\partial}{\partial \theta})$
$(e^{-2z}C^2)$	basic internal spherical coordinate operator in transformed coordinates $(e^{-2z}C^2) = -(2\frac{\partial^2}{\partial z \partial \theta} - \cot\theta \frac{\partial}{\partial z} + 3\frac{\partial}{\partial \theta})$
E^4	$(=E^2(E^2))$ (see Appendix D)
$\frac{\partial(f,g)}{\partial(x,y)}$	"Jacobian" transformation operator $\frac{\partial(f,g)}{\partial(x,y)} = \begin{vmatrix} \partial f/\partial x & \partial g/\partial y \\ \partial g/\partial x & \partial f/\partial y \end{vmatrix}$

$\frac{\partial}{\partial z}$	partial derivative with respect to z-direction
$\frac{\partial}{\partial \theta}$	partial derivative with respect to θ-direction

SUMMARY

This thesis examines the problem of the transient developing flow around a sphere moving within a Newtonian fluid in the low (non-creeping) Reynolds number region through computational techniques. First, the more general problem of two component direct contact heat and momentum transfer between continuous and disperse media is mathematically formulated via the fundamental vorticity-stream function approach. Furthermore, this work attempts to provide direction and a background for future research in this field. Finally, this work utilizes the formulation to demonstrate the initial solution of the developing transient isothermal flow around a solid sphere in motion in a fluid of infinite extent at low Reynolds numbers ($1.5 < Re < 40$).

CHAPTER I

INTRODUCTION

1.1. Motivation for the Research

Two-component direct-contact heat transfer with phase change is being intensively investigated at present, owing to its applicability to a wide variety of technical and industrial needs. Direct-contact heat transfer with phase change is the most efficient and optimal mode of heat transfer available for practical ranges of operating temperatures of many industrial operations ranging from steam power plants to space heating systems, and as such warrants continued investigation into its fundamental mechanisms. Investigations are needed into such specific processes as modeling the transport of momentum and energy associated with an immersed subcooled solid pellet heated through its liquid phase and ultimately expelled as a gas bubble from its heated fluid environment. Such an investigation demands a fundamental understanding of the complete discipline of heat transfer with phase change.

A very interesting recent investigation concerning the extraction of geothermal energy proposes to examine the thermodynamic cycle of working fluid isobutanol to remove the heat from water flooded into large fissures in the earth's hot crust. In this proposal, the isobutanol is sprayed in the form of liquid droplets at the bottom of these reservoirs of near boiling water, the droplets being at a temperature well below that of the water. These droplets are accelerated upward by buoyant forces as their bulk temperatures rise at very rapid rates. If designed properly, the system will allow these droplets to further absorb heat until slightly beyond the stage of saturated vapor into a superheated vapor state. In the process both vaporized isobutanol and small amounts of steam are subsequently expanded through a turbine. The expanded products are condensed and pumped back to the bottom of the fissure reservoirs. This innovative approach is a replica of traditional power plants with the exception that the fuel is the earth's heat and the "boiler" is the water holding reservoirs contained in the fissures.

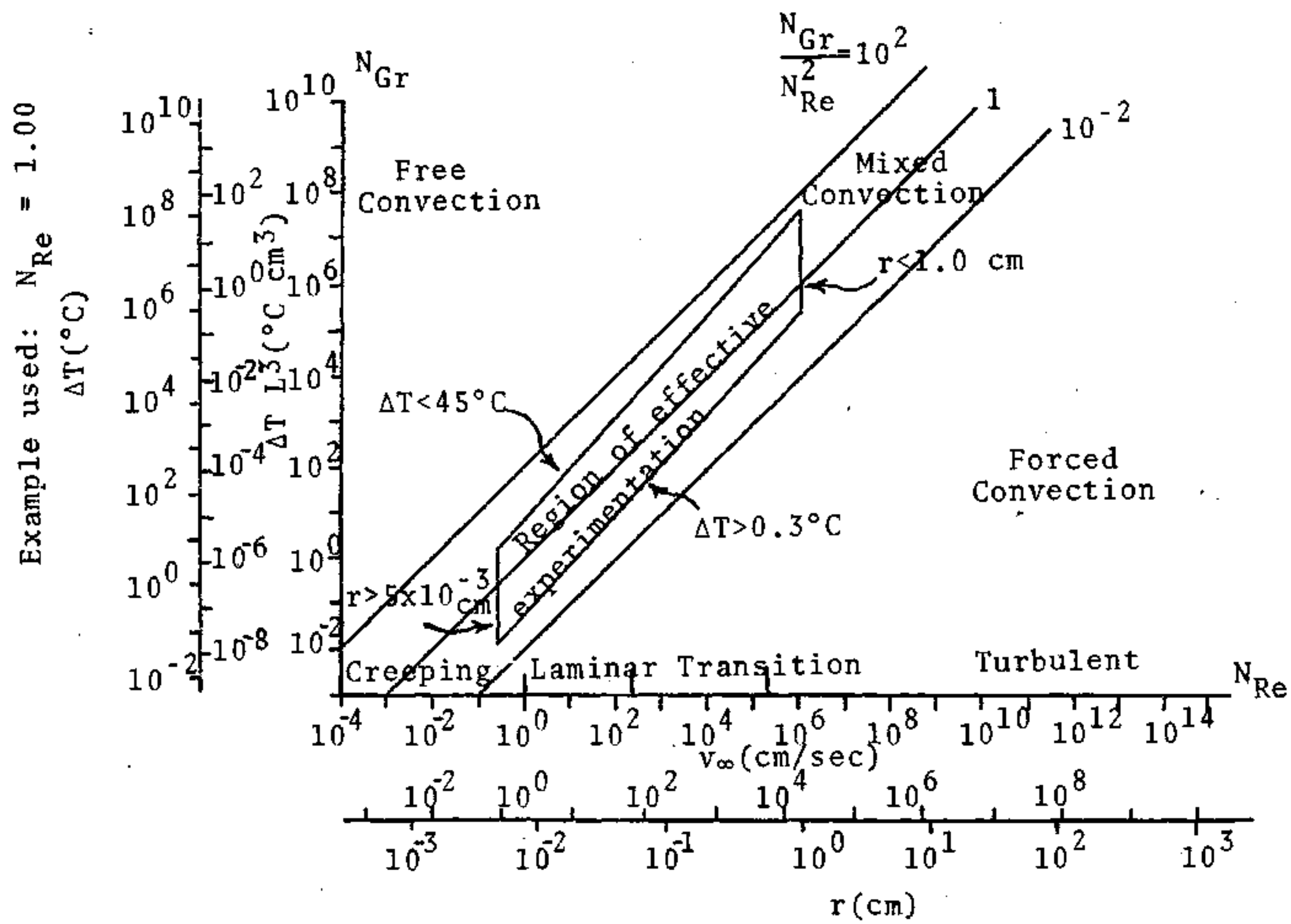
Prediction of yields, reliability, and operating limits of such a system also requires an extensive understanding of the complex interactions of two-component direct-contact heat transfer with phase change. However, at present, the only means of such predictions are through a vast collection of correlation equations, each limited to its region of applicability.

1.2. A Brief Review of Pertinent Literature

The field of transport phenomena around a spherical element suspended in an immiscible incompressible fluid medium started with the works of Stokes (1845, 1851), who introduced a special stream function to solve for the creeping motion of a sphere well over a century ago. Hadamard (1911, 1912) and Rybczynski (1911) further offered innovations to the problem by their viscosity parameter solutions for fluid spheres in creeping flow. Boussinesq (1905, 1913) also advanced the field through his perception of the significance of surface tension to the terminal settling velocity of fluid spheres. He also successfully demonstrated the effectiveness of assuming density to be a linear function of temperature. More recently Bassett (1961) developed the flow mechanism for the sedimentation of rigid spheres in the presence of a slip velocity due to sliding friction.

Hughes and Gilliland (1952) presented a very significant review of the concepts and correlations pertaining to terminal velocity investigations, and established guidelines on limits of terminal velocity based upon size of the settling body and the ratio of the internal and external fluid viscosities. Wallis (1974) published an excellent survey presenting an analysis of past terminal velocity research in the absence of heat transfer for both liquid-liquid and solid-liquid systems. He categorized various flow regions and brought into objective focus such factors as viscosity, surface tension, and surface contamination. He performed much of his work in the form of terminal velocity correlation equations. An alternative method of displaying his results is shown in Figure 1. The figure illustrates the region of effective experimentation for the heat transfer from water to a suspended light oil sphere at 21°C at terminal velocity. For example, in order to achieve a Reynolds number of unity, the fluid sphere must be about 0.064 millimeters in radius. The settling velocity will be about 0.244 centimeters per second, and for a practical limit of experimental measurement between 0.3°C and 45°C, the corresponding range of Grashof numbers is 0.03 through ten (10). Recently, Marchman and Sanford (1976) and Ueyama and Hatanaka (1976) have continued the investigation of terminal velocity through overall balances and integral concepts. However, most of the research during the past decade has been directed toward the topic of heat transfer of droplet sprays with phase change.

Jensen (1959) pioneered the ever expanding numerical analysis approach to the problem by introducing a successful finite-difference scheme for steady incompressible flow fields with respect to a radially stretched polar coordinate system. His method of incorporating pressure terms, drag, and other gross elements of overall interfacial behavior are still acceptable standards in such solutions. Hamielec and Johnson (1962) and Hamielec, Storey et al. (1963) used a power series technique to analytically solve the steady-state incompressible stream function equation without heat transfer. Schechter and Farley (1963a, 1963b) developed important guidelines for expressing the effects of surface



Example used: light oil sphere (21°C) suspended in water (93°C)

Figure 1. Region of Effective Experimentation for Liquid-Liquid System Based upon Wallis (1974)

tension upon the flow patterns surrounding spherical elements. They further demonstrated the importance of considering the exact stress equality boundary condition forms when solving for fluid-fluid systems where the ratio of the external viscosity and the internal viscosity was neither approaching zero nor infinite. Hamielec, Hoffman et al. (1967) and Hamielec, Johnson et al. (1967) investigated the accuracy and reliability of finite-difference solutions of the time-steady stream field for flow around spherical particles in intermediate Reynolds number range. Furthermore, they discussed appropriate boundary and initial conditions for the general problem of time-steady, viscous, incompressible, axisymmetric flow as well as the oscillating character of their finite-differencing scheme. However, they did not provide any recommendations on a technique for the solution of the time-unsteady, viscous, incompressible, axisymmetric flow problem. Roache (1972) compiled a very thorough survey and creative documentation on the many numbers of computational fluid dynamics approaches then available. His recommendations and notes on general finite-differencing techniques are very far-reaching, even though the text principally addresses itself to the rectangular and cylindrical coordinate systems. Much of the previous work, with more direct relevance to the present problem has involved both time-unsteady approaches to similar problems, as well as specific problems concerning time-steady, viscous, incompressible, axisymmetric flow with heat transfer.

In a promising investigation on combined forced and natural convective heat transfer, Pei (1969) correlated the Grashof number to the Reynolds number for solid-liquid systems. Sideman and Shabtai (1964) produced a most exhaustive survey on a large number of sources and solutions for direct-contact heat transfer between drops and surrounding liquid medium. Since then, however, newer works by Chao (1969) and Yao and Schrock (1976) have provided developments from fundamental theory that yield correlation equations within some constraints. Table 1 illustrates, in part, a survey by Sideman and Shabtai, in addition to some more recent investigations. For example, Sideman and Shabtai quoted an empirical formula for the continuous phase Nusselt number as

$$Nu_c = -178 + 3.62\, Re^{1/2} Pr^{1/3}, \qquad (1.1)$$

which applies to liquid drops with or without internal circulation for large Peclet numbers and intermediate Reynolds numbers ($50 < Re < 800$). Miura, Miura et al. (1976) also developed transport relations based on interactions between adjacently falling spheres. Harlow and Welch (1965) and Nakano and Tien (1967) provided excellent guidelines for time-dependent flow analysis of the stream function equation regardless of the coordinate system used. Payne (1958)

Table 1. Equations for Outside Nusselt Numbers for Fluid-Fluid Systems

Source	Nu_c	Re_c	Pr_c	Pe_c	Physical Description	Derivation
Sideman & Shabtai (1964)	$(\frac{12}{\pi})^{1/2} Pe^{1/2}$			Large	drop circulation	Simplified boundary layer theory
Sideman & Shabtai (1964)	$0.67\ Pe^{1/2} [\frac{\mu_c}{\mu_c+\mu_d}]^{1/2}$	<1		$\gg 2.4(3\frac{\mu_d}{\mu_c}+4)^2$ $(\frac{\mu_d}{\mu_c}+1)\gg 1$	drop circulation	Hadamand stream function
Sideman & Shabtai (1964)	$2+1.13\ Pe^{1/2} K_v^{1/2}$	>1	>>1		drop circulation	Potential flow
Sideman & Shabtai (1964)	$Nu_{rigid}(1-K_v)^{-1/2}$		1	Large	drop circulation	Reynolds analogy
Sideman & Shabtai (1964)	$a+g\ Pe^{1/2}$			Large	drop circulation	Empirical
Sideman & Shabtai (1964)	$-126+1.8\ Re^{1/2} Pr^{0.42}$			Large	drop circulation	Empirical
Sideman & Shabtai (1964)	$-178+3.62\ Re^{1/2} Pr^{1/3}$	50-800		Large	drop with and without circulation	Empirical

Table 1 (concluded)

Sideman & Taitel (1964)	$Nu=\left(\frac{3\cos\beta-\cos^3\beta+2}{\pi}\right)^{1/2}Pe^{1/2}$	>1000	>1	>5500	Drop without internal circulation	Potential flow
Chao (1969)	$\frac{2\sqrt{Pe}}{\sqrt{\pi}\ (1+\beta)}-I$	Large		Large	Drop circulation inviscid	Thin boundary layer
Somer, Bora et al. (1973)	$(2x10^{-4})\,Pe^{1.3}$	2-60			Drop, with and without circulation	Empirical
Yao & Schrock (1976)	$2+15\left(\frac{x}{d}\right)^{-0.7}Pr^{1/3}Re^{1/2}$					Ranz-Marshall correlation

published his solution for transient, viscous flow around a cylinder, and many of his concepts in time-unsteady finite-differencing and data previewing are noteworthy.

Scriven (1960), Scriven and Sternling (1960), and Miller and Scriven (1968) discuss the most general formulations available on the dynamics of a Newtonian fluid interface. These works present the topic of boundary conditions with respect to static, dynamic, and oscillatory rheological behavior.

In short, the interest in the transient, viscous, incompressible flow around fluid spheres with heat transfer has been sustained in recent years due to a wide variety of its possible industrial applications. However, the previous efforts, individually or collectively, have not investigated the problem in its entirety. Major advances are needed into the investigation of the transient flow field development around a sphere immersed in a viscous, incompressible fluid in the presence of heat transfer. The general equations of motion and transport must be developed for fluids with variable properties. The important boundary and initial conditions must be fully understood and their influence on each application needs to be illuminated.

1.3. Statement of the Research Problem

When considering problems of transient direct contact heat transfer, there appear to be three major classifications of characteristic dynamic fluid behavior. Figure 2 illustrates these three classes. Problems of class I are typical of fluids with extremely low Prandtl numbers, where the temperature of the internal fluid reaches the ambient temperature much more quickly than the sphere reaches terminal velocity. Problems of class II are typical of fluids with extremely high Prandtl numbers, where the terminal velocity is reached almost instantaneously compared with the time required for the sphere to reach ambient temperature. Problems of class III represent a more realistic fluid with an intermediate Prandtl number, with characteristic times to reach ultimate temperature and velocity being of the same relative magnitude. It may be observed that the majority of the investigations available fall into one of the last two regions of the problems of class II. In other words, these investigations presume the terminal velocity mechanism at the inception of their analyses, and fail to comment upon the transient velocity as it develops. It is also interesting to note, for example, that for problems of class III, beyond some critical time, t_c, the fluid sphere's acceleration is no longer due to the traditional buoyancy force. The fluid sphere is accelerated by its own volumetric changes being driven by the sustained heat transfer. For carefully controlled experiments, it may be possible to observe a virtual terminal settling velocity, which will not be constant. As the fluid sphere gets heated its density will decrease

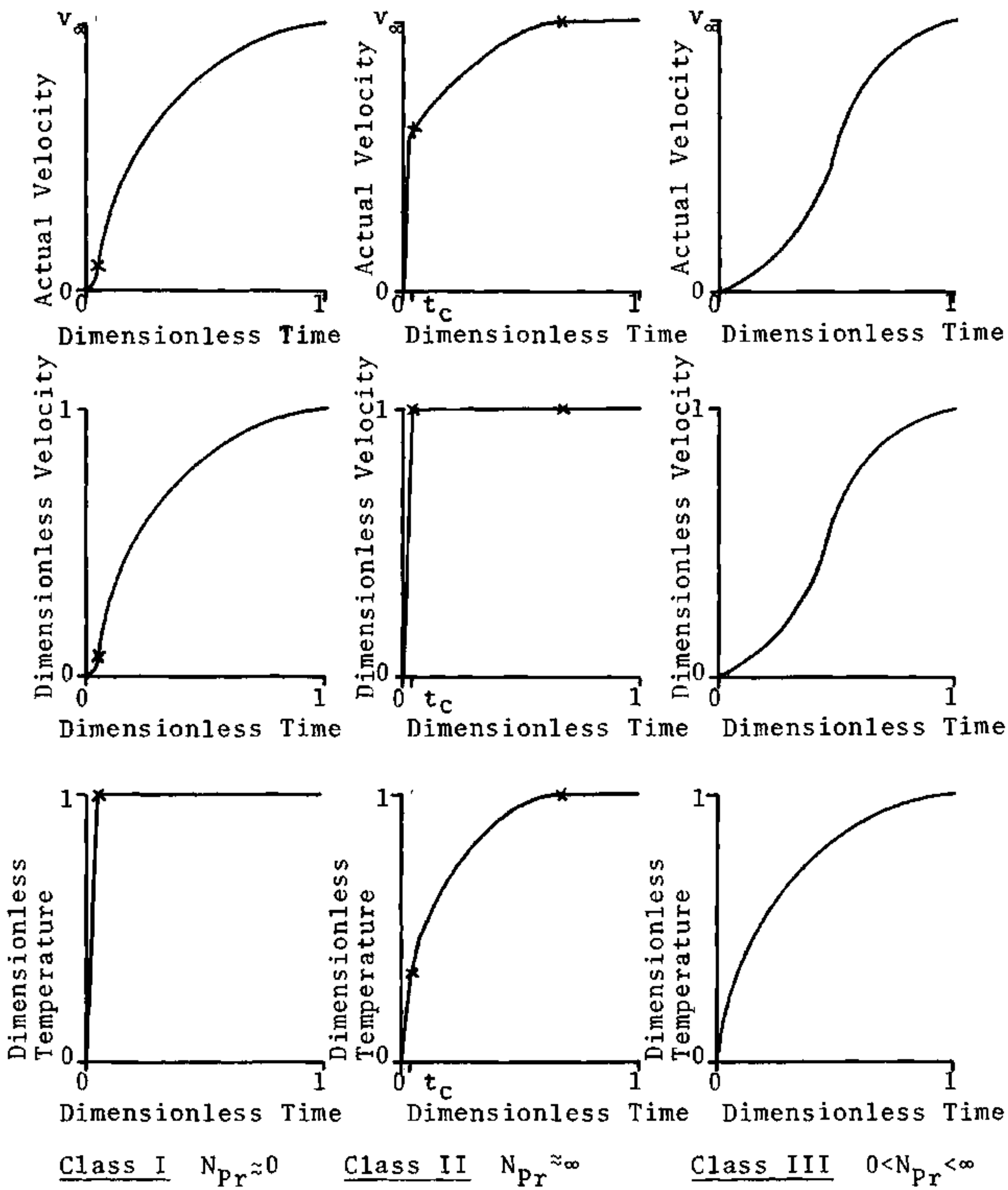


Figure 2. Classifications of Unsteady Heat Transfer from Fluid Droplets with Variable Fluid Properties

instantaneously, and consequently it displaces a slightly higher amount of its surrounding fluid. Even though the gravitational, buoyancy, and drag forces balanced just a moment before, this new infinitesimal change in volume immediately demands a new force equilibrium, thus requiring the fluid sphere to accelerate until the new drag force reestablishes the equilibrium. The virtual terminal settling velocity phenomenon, though not actually verified experimentally, has a definite theoretic basis.

Though much enthusiasm is present in the field of direct-contact heat transfer, little research has been accomplished relative to the desired end result of transient direct-contact heat transfer. Questions such as the nature of the transient flow field around a solid or liquid sphere as it settles from rest in another fluid of infinite extent remain totally unanswered. The extent to which general solutions of transient direct-contact heat transfer with phase change may be applied to real industrial problems will remain unknown until such investigations are initiated. This work attempts to provide direction and a background for future research in this field by mathematically formulating the problem of heat and momentum transport mechanisms between continuous and disperse media. It is further intended to utilize the formulation to demonstrate the initial solution of the developing transient isothermal flow around a solid sphere in non-creeping motion in a fluid of infinite extent at low Reynolds numbers ($1.5 < Re < 40$).

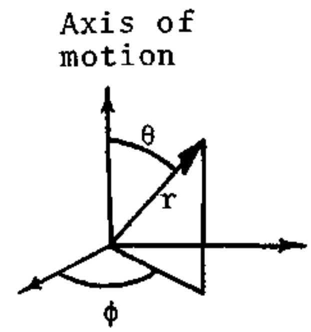
CHAPTER II

MATHEMATICAL FORMULATION

2.1. The Differential Analysis Approach

The primary scope of this general development is to build a system of equations which will provide a stable solution for velocity, temperature, and stream function profiles in space as functions of time for a closed range of fluid or solid spheres in suspended motion in a fluid of infinite extent. Though it is possible to provide estimates on the mean sphere velocity and temperature as time proceeds for special cases of inner fluid behavior by using overall control volume analysis, this technique alone is insufficient to provide the required solution for all cases of inner fluid behavior. For example, inner fluids which exhibit variable surface tension on the surface will greatly complicate the dynamic boundary conditions beyond the scope of overall control volume analysis. Also, in analyzing fluid sphere motions, where the ratio of viscosities of the inner and outer fluids is neither too small nor too large to allow assymptotic approximations, the integral control volume approach proves to be totally inadequate.

The problem of developing the appropriate equations of transport for a spherical fluid element immiscibly suspended in another liquid is somewhat complex. The best choice of coordinates is strongly suggested by the physical character of the element itself. Because there is every reason to assume that the element cannot maintain a spin in any conventional application, a spherical coordinate system is chosen with complete ϕ-independence. It will be assumed throughout the work that all equations of transport apply to the spherical (r,θ,ϕ) coordinate system as illustrated at right.



Spherical coordinate system

The major assumptions adopted in the analysis are:

(1) Spherical coordinate system.

(2) Axisymmetric character (no dependency upon ϕ-component in equations or fluid properties).

(3) Unsteady character with respect to time.

(4) Viscous fluids.

(5) Variable fluid properties (ρ,μ,k,c_v) as functions of temperature.

(6) Negligible viscous dissipation.

(7) $\nabla\cdot\vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta \sin\theta) \equiv 0$ along a streamline.

(8) Spherical element maintains spherical shape.

2.2. Development of the Transport Equations

The basic equations of momentum and heat transport for axisymmetric flow in a spherical frame are:

(a) The dimensional continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 v_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\rho v_\theta \sin\theta) = 0, \tag{2.1}$$

(b) The dimensional momentum equations

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} \tag{2.2}$$

$$-\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{rr}) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\tau_{r\theta}\sin\theta) - \frac{\tau_{\theta\theta}+\tau_{\phi\phi}}{r}\right) + \rho g_r, \text{ and}$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} \tag{2.3}$$

$$-\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\tau_{\theta\theta}\sin\theta) + \frac{\tau_{r\theta}-\cot\theta\,\tau_{\phi\phi}}{r}\right) + \rho g_\theta,$$

and

(c) The dimensional energy equation

$$\rho c_v\left(\frac{\partial T}{\partial t} + v_r\frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta}\right) = -\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 q_r) + \right. \tag{2.4}$$

$$\left.\frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(q_\theta \sin\theta)\right) - T\left(\frac{\partial p}{\partial T}\right)_\rho (\nabla\cdot\vec{V}) + \mu\Phi_v.$$

For a Newtonian incompressible fluid with temperature dependent properties, flowing with negligible viscous dissipation, the transport equations become

(a) The dimensional continuity equation

$$\left(\frac{\partial \rho}{\partial t} + v_r \frac{\partial \rho}{\partial r} + \frac{v_\theta}{r} \frac{\partial \rho}{\partial \theta}\right) + \rho(\nabla \cdot \vec{V}) = 0, \qquad (2.5)$$

(b) The dimensional momentum equations

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r}\right) = - \frac{\partial p}{\partial r} + \rho g_r \qquad (2.6)$$

$$+ \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 2\mu \frac{\partial v_r}{\partial r}\right) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} \left(\mu r \sin\theta \frac{\partial}{\partial r} \left(\frac{v_\theta}{r}\right) + \right.\right.$$

$$\left.\left.\frac{\mu}{r} \sin\theta \frac{\partial v_r}{\partial \theta}\right) - \frac{2\mu}{r} \left(2v_r + \frac{\partial v_\theta}{\partial \theta} + v_\theta \cot\theta\right)\right), \text{ and}$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r}\right) = - \frac{1}{r} \frac{\partial p}{\partial \theta} + \rho g_\theta \qquad (2.7)$$

$$+ \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(\mu r^3 \frac{\partial}{\partial r} \left(\frac{v_\theta}{r}\right) + \mu r \frac{\partial v_r}{\partial \theta}\right) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} \left(2\mu \sin\theta \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)\right)\right.$$

$$\left. + \mu\left(\frac{\partial}{\partial r} \left(\frac{v_\theta}{r}\right) + \frac{1}{r^2} \frac{\partial v_r}{\partial \theta}\right) - \frac{2\mu}{r^2} (v_r + v_\theta \cot\theta) \cot\theta\right), \text{ and}$$

(c) The dimensional energy equation

$$\rho c_v \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta}\right) = \tag{2.8}$$

$$\left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 k \frac{\partial T}{\partial r}\right) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} \left(k \frac{\sin\theta}{r} \frac{\partial T}{\partial \theta}\right)\right).$$

The incompressible flow continuity equation is automatically satisfied by introducing the stream function, ψ, defined by

$$v_r = - \frac{1}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta}; \quad v_\theta = \frac{1}{r \sin\theta} \frac{\partial \psi}{\partial r}. \tag{2.9}$$

Cross differentiation of the radial and tangential components of the momentum equation to eliminate the pressure terms, yields, for a variable property fluid, the stream function equation (see Appendix D)

$$\frac{1}{\sin\theta} \left(\frac{\partial^2 \psi}{\partial t\, \partial r} \frac{\partial \rho}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial t\, \partial \theta} \frac{\partial \rho}{\partial \theta}\right) + \frac{\rho}{\sin\theta} \frac{\partial}{\partial t} \left(E^2 \psi\right) + \rho \frac{\partial\left(\psi, \frac{E^2 \psi}{r^2 \sin^2\theta}\right)}{\partial (r,\theta)}$$

(2.10)

$$- \frac{1}{r \sin^2\theta} \left[\left(\frac{1}{r^2} \frac{\partial^2 \psi}{\partial r\, \partial \theta} \frac{\partial \psi}{\partial \theta} - \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \frac{\partial \psi}{\partial r} - \frac{1}{r} \left(\frac{\partial \psi}{\partial r}\right)^2 + \frac{\cot\theta}{r^2} \frac{\partial \psi}{\partial r} \frac{\partial \psi}{\partial \theta} - \right.\right.$$

$$\left.\frac{2}{r^3} \left(\frac{\partial \psi}{\partial \theta}\right)^2\right) \frac{1}{r} \frac{\partial \rho}{\partial \theta} + \left(\frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} \frac{\partial \psi}{\partial \theta} - \frac{1}{r} \frac{\partial^2 \psi}{\partial r\, \partial \theta} \frac{\partial \psi}{\partial r} + \frac{\cot\theta}{r} \left(\frac{\partial \psi}{\partial r}\right)^2\right) \frac{\partial \rho}{\partial r}\Big]$$

Equation (2.10) continued

$$= \frac{\mu}{\sin\theta} E^4\psi + \frac{1}{\sin\theta} [2 \frac{\partial}{\partial r} (E^2\psi - \frac{1}{r} \frac{\partial\psi}{\partial r})] \frac{\partial\mu}{\partial r} + \frac{1}{\sin\theta} [\frac{2}{r^2} \frac{\partial}{\partial\theta} (E^2\psi) -$$

$$\frac{\cot\theta}{r^2} (E^2\psi) - \frac{2}{r^2} (C^2\psi)] \frac{\partial\mu}{\partial\theta} + \frac{1}{\sin\theta} [2r \frac{\partial}{\partial r} (\frac{1}{r} \frac{\partial\psi}{\partial r}) -$$

$$E^2\psi](\frac{\partial^2\mu}{\partial r^2} - \frac{1}{r^2} \frac{\partial^2\mu}{\partial\theta^2}) + \frac{1}{\sin\theta} [\frac{2}{r} (C^2\psi)] \frac{\partial^2\mu}{\partial r\partial\theta}. \qquad (2.10)$$

Also, the stream function relations may be incorporated into the energy equation to obtain

$$\rho c_v [\frac{\partial T}{\partial t} - \frac{1}{r^2\sin\theta} \frac{\partial(T,\psi)}{\partial(r,\theta)}] = k\nabla^2 T + (\nabla k)\cdot(\nabla T). \qquad (2.11)$$

Nondimensionalization of the energy and stream function equations is achieved by a set of dimensionless variables and operators (each shown by a superscript) as

$$\rho^+ = \frac{\rho}{\rho_o} \qquad r^+ = \frac{r}{r_o} \qquad \nabla^+ = r_o\nabla \qquad (2.12)$$

$$c_v^+ = \frac{c_v}{c_{v_o}} \qquad \psi^+ = \frac{\psi}{Ur_o^2} \qquad E^{2^+} = r_o^2E^2$$

$$k^+ = \frac{k}{k_o} \qquad t^+ = \frac{U}{r_o}t \qquad E^{2^+}\psi^+ = \frac{1}{U} E^2\psi$$

$$T^+ = \frac{T-T_o}{T_\infty-T_o} \qquad \mu^+ = \frac{\mu}{\mu_o} \qquad C^{2+} = r_o^2C^2$$

Equation (2.12) continued

$$N_{Re} = 2\,\frac{\rho_o U r_o}{\mu_o} \qquad p^+ = \frac{p}{\frac{2\mu_o U}{r_o}} \qquad C^{2^+}\psi^+ = \frac{1}{U}\,c^2\psi$$

$$N_{Pr} = \frac{\mu_o c p_o}{k_o} \qquad \sigma^+ = \frac{\sigma}{\mu_o U}\ . \tag{2.12}$$

A substitution of the dimensionless quantities into the dimensional equations of transport render the normalized set of equations (with superscripts excluded for convenience), comprising of

(a) The dimensionless variable property stream function equation

$$\frac{N_{Re}}{2}\{(\frac{\partial^2\psi}{\partial t\partial r}\,\frac{\partial\rho}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2\psi}{\partial t\partial\theta}\,\frac{\partial\rho}{\partial\theta}) + \rho\,\frac{\partial}{\partial t}\,(E^2\psi) - \rho\,\frac{\partial(\frac{E^2\psi}{r^2\sin^2\theta},\psi)}{\partial(r,\theta)}\sin\theta\} \tag{2.13}$$

$$- \frac{N_{Re}}{2}\,\frac{1}{r\sin\theta}\,[(\frac{1}{r^2}\,\frac{\partial^2\psi}{\partial r\partial\theta}\,\frac{\partial\psi}{\partial\theta} - \frac{1}{r^2}\,\frac{\partial^2\psi}{\partial\theta^2}\,\frac{\partial\psi}{\partial r} - \frac{1}{r}\,(\frac{\partial\psi}{\partial r})^2 +$$

$$\frac{\cot\theta}{r^2}\,\frac{\partial\psi}{\partial r}\,\frac{\partial\psi}{\partial\theta} - \frac{2}{r^3}\,(\frac{\partial\psi}{\partial\theta})^2)\,\frac{1}{r}\,\frac{\partial\rho}{\partial\theta} + (\frac{1}{r}\,\frac{\partial^2\psi}{\partial r^2}\,\frac{\partial\psi}{\partial\theta} - \frac{1}{r}\,\frac{\partial^2\psi}{\partial r\partial\theta}\,\frac{\partial\psi}{\partial r} +$$

$$\frac{\cot\theta}{r}\,(\frac{\partial\psi}{\partial r})^2)\,\frac{\partial\rho}{\partial r}]\}$$

$$= \mu E^4\psi + [2\,\frac{\partial}{\partial r}\,(E^2\psi - \frac{1}{r}\,\frac{\partial\psi}{\partial r})]\,\frac{\partial\mu}{\partial r} + [\frac{2}{r^2}\,\frac{\partial}{\partial\theta}\,(E^2\psi) -$$

Equation (2.13) continued

$$\frac{\cot\theta}{r^2}(E^2\psi) - \frac{2}{r^2}(C^2\psi)]\frac{\partial\mu}{\partial\theta} + [2r\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial\psi}{\partial r}) - E^2\psi](\frac{\partial^2\mu}{\partial r^2} -$$

$$\frac{1}{r^2}\frac{\partial^2\mu}{\partial\theta^2}) + [\frac{2}{r}(C^2\psi)]\frac{\partial^2\mu}{\partial r\partial\theta}, \text{ and} \tag{2.13}$$

(b) The dimensionless variable property energy equation

$$\frac{N_{Re}}{2}N_{Pr}\{\rho c_v[\frac{\partial T}{\partial t} - \frac{1}{r^2\sin\theta}\frac{\partial(T,\psi)}{\partial(r,\theta)}]\} = k(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \tag{2.14}$$

$$\frac{1}{r^2}\frac{\partial^2 T}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial T}{\partial\theta}) + (\frac{\partial k}{\partial r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial k}{\partial\theta}\frac{\partial T}{\partial\theta}).$$

It is immediately observed that Equation (2.13) is a highly non-linear fourth-order partial differential equation, and as such might not prove directly solvable. However, by introducing the vorticity function, ζ, defined by (see Appendix C)

$$E^2\psi = \zeta r\sin\theta, \tag{2.15}$$

Equation (2.13) may be reduced to two simultaneous second-order partial differential equations in ψ, the latter to be solved by numerical methods.

As has been pointed out by Jensen (1959), near the surface of the fluid spherical interface it is expected that the stream function and the whole character of each equation will vary most radically. Hence it is desirable to use, for relaxation purposes, a fine lattice near the interface, and to allow the lattice to become increasingly more coarse in the region farther removed from the interface. This may be accomplished quite elegantly by the stretching transformations $r = e^{z}$ for regular intervals of z for the external fluid, and $r = e^{-z}$ for regular intervals of z for the internal fluid. Furthermore, to separate the stream function equation into two solvable representations the introduction of several new variables is necessary. These are represented for the external and the internal flows as follows:

(a) External fluid ($r = e^{z}$)

$$\zeta = \frac{1}{e^{3z}\sin\theta}\,(e^{2z}E^2)\psi \tag{2.16a}$$

$$F = \frac{\zeta}{r\sin\theta} = \frac{\zeta}{e^{z}\sin\theta} = \frac{1}{e^{4z}\sin^2\theta}\,(e^{2z}E^2)\psi$$

$$G = \zeta r\sin\theta = \zeta e^{z}\sin\theta = \frac{1}{e^{2z}}\,(e^{2z}E^2)\psi$$

$$(e^{2z}E^2)\psi = \left(\frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial z} + \frac{\partial^2}{\partial\theta^2} - \cot\theta\,\frac{\partial}{\partial\theta}\right)\psi, \text{ and}$$

(b) Internal fluid ($r = e^{-z}$)

$$\zeta = \frac{e^{3z}}{\sin\theta}\,(e^{-2z}E^2)\psi \tag{2.16b}$$

$$F = \frac{\zeta}{r\sin\theta} = \frac{e^{z}}{\sin\theta}\,\zeta = \frac{e^{4z}}{\sin^2\theta}\,(e^{-2z}E^2)\psi$$

$$G = \zeta r\sin\theta = \frac{\sin\theta}{e^{z}}\,\zeta = e^{2z}(e^{-2z}E^2)\psi$$

$$(e^{-2z}E^2)\psi = \left(\frac{\partial^2}{\partial z^2} + \frac{\partial}{\partial z} + \frac{\partial^2}{\partial\theta^2} - \cot\theta\,\frac{\partial}{\partial\theta}\right)\psi.$$

Use of these variables into the equations of transport for the internal fluid yields

(a) The internal dimensionless variable properties vorticity-stream function equation

$$\frac{N_{Re}}{2}\left\{e^{2z}\left(\frac{\partial^2\psi}{\partial t\partial z}\,\frac{\partial\rho}{\partial z} + \frac{\partial^2\psi}{\partial t\partial\theta}\,\frac{\partial\rho}{\partial\theta}\right) + \rho\,\frac{\partial}{\partial t}\,(G) + \rho e^{z}\,\frac{\partial(F,\psi)}{\partial(z,\theta)}\,\sin\theta\right\} \tag{2.17}$$

$$+ \frac{N_{Re}}{2}\left\{\frac{e^{5z}}{\sin\theta}\left[\left(\frac{\partial^2\psi}{\partial z\partial\theta}\,\frac{\partial\psi}{\partial\theta} - \frac{\partial^2\psi}{\partial\theta^2}\,\frac{\partial\psi}{\partial z} + \left(\frac{\partial\psi}{\partial z}\right)^2 + \cot\theta\,\frac{\partial\psi}{\partial z}\,\frac{\partial\psi}{\partial\theta} + \right.\right.\right.$$

$$\left.\left.\left.\left(\frac{\partial\psi}{\partial\theta}\right)^2\right)\frac{\partial\rho}{\partial\theta} + \left(\frac{\partial^2\psi}{\partial z^2}\,\frac{\partial\psi}{\partial\theta} + \frac{\partial\psi}{\partial z}\,\frac{\partial\psi}{\partial\theta} - \frac{\partial^2\psi}{\partial z\partial\theta}\,\frac{\partial\psi}{\partial z} + \cot\theta\,\left(\frac{\partial\psi}{\partial z}\right)^2\right)\frac{\partial\rho}{\partial z}\right]\right\}$$

$$= \mu e^{2z}((e^{-2z}E^2)(G)) + e^{2z}\left[2\left(\frac{\partial G}{\partial z} + e^{2z}\,\frac{\partial^2\psi}{\partial z^2} + 2e^{2z}\,\frac{\partial\psi}{\partial z}\right)\right]\frac{\partial\mu}{\partial z} +$$

Equation (2.17) continued

$$+ e^{2z}[2\frac{\partial G}{\partial\theta} - (\cot\theta)G + 2e^{2z}(e^{-2z}C^2)\psi]\frac{\partial\mu}{\partial\theta} + e^{2z}[2e^{2z}(\frac{\partial^2\psi}{\partial z^2} + 2\frac{\partial\psi}{\partial z}) - G]$$

$$(\frac{\partial^2\mu}{\partial z^2} + \frac{\partial\mu}{\partial z} - \frac{\partial^2\mu}{\partial\theta^2}) + e^{2z}[2e^{2z}(e^{-2z}C^2)\psi]\frac{\partial^2\mu}{\partial z\partial\theta}, \text{ and} \qquad (2.17)$$

(b) The internal dimensionless variable properties energy equation

$$\frac{N_{Re}}{2} N_{Pr}\{\rho c_v[\frac{\partial T}{\partial t} + \frac{e^{3z}}{\sin\theta}\frac{\partial(T,\psi)}{\partial(z,\theta)}]\} = e^{2z}[k(\frac{\partial^2 T}{\partial z^2} - \frac{\partial T}{\partial z} + \frac{\partial^2 T}{\partial\theta^2} + \qquad (2.18)$$

$$\cot\theta\frac{\partial T}{\partial\theta}) + (\frac{\partial k}{\partial z}\frac{\partial T}{\partial z} + \frac{\partial k}{\partial\theta}\frac{\partial T}{\partial\theta})].$$

The corresponding equations of transport for the external fluid are

(a) The external dimensionless variable properties vorticity-stream function equation

$$\frac{N_{Re}}{2}\{e^{-2z}(\frac{\partial^2\psi}{\partial t\partial z}\frac{\partial\rho}{\partial z} + \frac{\partial^2\psi}{\partial t\partial\theta}\frac{\partial\rho}{\partial\theta}) + \rho\frac{\partial}{\partial t}(G) - \rho e^{-z}\frac{\partial(F,\psi)}{\partial(z,\theta)}\sin\theta\} \qquad (2.19)$$

$$-\frac{N_{Re}}{2}\frac{e^{-5z}}{\sin\theta}[(\frac{\partial^2\psi}{\partial z\partial\theta}\frac{\partial\psi}{\partial\theta} - \frac{\partial^2\psi}{\partial\theta^2}\frac{\partial\psi}{\partial z} - (\frac{\partial\psi}{\partial z})^2 + \cot\theta\frac{\partial\psi}{\partial z}\frac{\partial\psi}{\partial\theta} - 2(\frac{\partial\psi}{\partial\theta})^2)\frac{\partial\rho}{\partial\theta}$$

Equation (2.19) continued

$$+ \left(\frac{\partial^2\psi}{\partial z^2}\frac{\partial\psi}{\partial\theta} - \frac{\partial\psi}{\partial z}\frac{\partial\psi}{\partial\theta} - \frac{\partial^2\psi}{\partial z\partial\theta}\frac{\partial\psi}{\partial z} + \cot\theta\left(\frac{\partial\psi}{\partial z}\right)^2\right)\frac{\partial\rho}{\partial z}]\}$$

$$= \mu e^{-2z}((e^{2z}E^2)(G)) + e^{-2z}[2(\frac{\partial G}{\partial z} - e^{-2z}\frac{\partial^2\psi}{\partial z^2} + 2e^{-2z}\frac{\partial\psi}{\partial z})]\frac{\partial\mu}{\partial z}$$

$$+ e^{-2z}[2\frac{\partial G}{\partial\theta} - (\cot\theta)G - 2e^{-2z}(e^{2z}C^2)\psi]\frac{\partial\mu}{\partial\theta}$$

$$+ e^{-2z}[2e^{-2z}(\frac{\partial^2\psi}{\partial z^2} - 2\frac{\partial\psi}{\partial z}) - G](\frac{\partial^2\mu}{\partial z^2} - \frac{\partial\mu}{\partial z} - \frac{\partial^2\mu}{\partial\theta^2})$$

$$+ e^{-2z}[2e^{-2z}(e^{2z}C^2)\psi]\frac{\partial^2\mu}{\partial z\partial\theta}, \text{ and} \tag{2.19}$$

(b) The external dimensionless variable properties energy equation

$$\frac{N_{Re}}{2}N_{Pr}\{\rho c_v[\frac{\partial T}{\partial t} - \frac{e^{-3z}}{\sin\theta}\frac{\partial(T,\psi)}{\partial(z,\theta)}]\} = e^{-2z}[k(\frac{\partial^2 T}{\partial z^2} + \frac{\partial T}{\partial z} + \frac{\partial^2 T}{\partial\theta^2} +$$

$$\cot\theta\frac{\partial T}{\partial\theta}) + (\frac{\partial k}{\partial z}\frac{\partial T}{\partial z} + \frac{\partial k}{\partial\theta}\frac{\partial T}{\partial\theta})]. \tag{2.20}$$

Finally, the last series of substitutions to be made in order to facilitate computational aspects of the equations are approximations to the various fluid properties as functions of temperature (assuming constant pressure). Following Boussinesq (1905, 1913) the density, thermal conductivity, and constant volume specific heat may be respectively approximated as

$$\rho = 1-\beta T \qquad \frac{d\rho}{dT} = -\beta \tag{2.21a}$$

$$k = 1-\gamma T \qquad \frac{dk}{dT} = -\gamma$$

$$c_v = 1-\varepsilon T$$

The viscosity dependence upon temperature is sufficiently strong, according to Torrance and Turcotte (1971), to be approximated by the exponential function

$$\mu = e^{-\lambda T} \qquad \frac{d\mu}{dT} = -\lambda e^{-\lambda T} \qquad \frac{d^2\mu}{dT^2} = \lambda^2 e^{-\lambda T} \tag{2.21b}$$

Schechter and Farley (1963a, 1963b) suggested that fluid impurities cause a variation in surface tension with the angular position along the droplet surface which may be approximated by a cosine function. In addition, the surface tension is expected to vary with sphere radius and temperature. It may thus be approximated by the expression

$$\sigma = \delta(T)(1-\chi(T)\cos\theta), \tag{2.22}$$

where $\delta(T)$ is the bulk surface tension acting between the two interfacial fluids and $\chi(T)$ is the correlation coefficient to account for the variable concentration of surface contamination from the forward to the rear stagnation points.

With these substitutions in force, the final forms of the equations of transport strongly coupled in $\psi(z,\theta,t)$ and $T(z,\theta,t)$ for the internal (disperse) and external (continuous) flows are:

(A) Internal fluid

(a) Dimensionless variable properties vorticity-stream function equation

$$
\begin{aligned}
&\frac{N_{Re}}{2}\{-\beta e^{2z}(\frac{\partial^2\psi}{\partial t\partial z}\,\frac{\partial T}{\partial z} + \frac{\partial^2\psi}{\partial t\partial\theta}\,\frac{\partial T}{\partial\theta}) + (1-\beta T)\frac{\partial}{\partial t}(G) + (1-\beta T)e^{z}\,\frac{\partial(F,\psi)}{\partial(z,\theta)}\,\sin\theta\} \qquad (2.23)\\
&-\beta\,\frac{N_{Re}}{2}\,\{\frac{e^{5z}}{\sin\theta}\,[(\frac{\partial^2\psi}{\partial z\partial\theta}\,\frac{\partial\psi}{\partial\theta} - \frac{\partial^2\psi}{\partial\theta^2}\,\frac{\partial\psi}{\partial z} + (\frac{\partial\psi}{\partial z})^2 + \cot\theta\,\frac{\partial\psi}{\partial z}\,\frac{\partial\psi}{\partial\theta}\\
&+ (\frac{\partial\psi}{\partial\theta})^2)\,\frac{\partial T}{\partial\theta} + (\frac{\partial^2\psi}{\partial z^2}\,\frac{\partial\psi}{\partial\theta} + \frac{\partial\psi}{\partial z}\,\frac{\partial\psi}{\partial\theta} - \frac{\partial^2\psi}{\partial z\partial\theta}\,\frac{\partial\psi}{\partial z} + \cot\theta\,(\frac{\partial\psi}{\partial z})^2)\,\frac{\partial T}{\partial z}]\}\\
&= e^{-\lambda T}\{e^{2z}((e^{-2z}E^2)(G)) - \lambda e^{2z}[2(\frac{\partial G}{\partial z} + e^{2z}\,\frac{\partial^2\psi}{\partial z^2} + 2e^{2z}\,\frac{\partial\psi}{\partial z})]\,\frac{\partial T}{\partial z}\\
&- \lambda e^{2z}[2\,\frac{\partial G}{\partial\theta} - (\cot\theta)G + 2e^{2z}(e^{-2z}C^2)\psi]\,\frac{\partial T}{\partial\theta}\\
&- \lambda e^{2z}[2e^{2z}(\frac{\partial^2\psi}{\partial z^2} + 2\,\frac{\partial\psi}{\partial z}) - G](\frac{\partial^2 T}{\partial z^2} - \lambda\,(\frac{\partial T}{\partial z})^2 + \frac{\partial T}{\partial z} - \frac{\partial^2 T}{\partial\theta^2} + \lambda(\frac{\partial T}{\partial\theta})^2)\\
&- \lambda e^{2z}[2e^{2z}(e^{-2z}C^2)\psi](\frac{\partial^2 T}{\partial z\partial\theta} - \lambda\,\frac{\partial T}{\partial z}\,\frac{\partial T}{\partial\theta}).
\end{aligned}
$$

(b) Dimensionless variable properties energy equation

$$\frac{N_{Re}}{2} N_{Pr}\{(1-\beta T)(1-\varepsilon T)[\frac{\partial T}{\partial t} + \frac{e^{3z}}{\sin\theta} \frac{\partial(T,\psi)}{\partial(z,\theta)}]\} = \qquad (2.24)$$

$$e^{2z}[(1-\gamma T)(\frac{\partial^2 T}{\partial z^2} - \frac{\partial T}{\partial z} + \frac{\partial^2 T}{\partial \theta^2} + \cot\theta \frac{\partial T}{\partial \theta}) - \gamma((\frac{\partial T}{\partial z})^2 + (\frac{\partial T}{\partial \theta})^2)].$$

(B) External fluid

(a) Dimensionless variable properties vorticity-stream function equation

$$\frac{N_{Re}}{2} \{-\beta e^{-2z}(\frac{\partial^2\psi}{\partial t\partial z} \frac{\partial T}{\partial z} + \frac{\partial^2\psi}{\partial t\partial\theta} \frac{\partial T}{\partial\theta}) + (1-\beta T)\frac{\partial}{\partial t}(G) - (1-\beta T)e^{-z} \frac{\partial(F,\psi)}{\partial(z,\theta)}\sin\theta\} \qquad (2.25)$$

$$+\beta \frac{N_{Re}}{2}\{\frac{e^{-5z}}{\sin\theta} [(\frac{\partial^2\psi}{\partial z\partial\theta} \frac{\partial\psi}{\partial\theta} - \frac{\partial^2\psi}{\partial\theta^2} \frac{\partial\psi}{\partial z} - (\frac{\partial\psi}{\partial z})^2 + \cot\theta \frac{\partial\psi}{\partial z} \frac{\partial\psi}{\partial\theta}$$

$$-2(\frac{\partial\psi}{\partial\theta})^2) \frac{\partial T}{\partial\theta} + (\frac{\partial^2\psi}{\partial z^2} \frac{\partial\psi}{\partial\theta} - \frac{\partial\psi}{\partial z} \frac{\partial\psi}{\partial\theta} - \frac{\partial^2\psi}{\partial z\partial\theta} \frac{\partial\psi}{\partial z} + \cot\theta (\frac{\partial\psi}{\partial z})^2) \frac{\partial T}{\partial z}]\}$$

$$= e^{-\lambda T}\{e^{-2z}((e^{2z}E^2)(G)) - \lambda e^{-2z}[2(\frac{\partial G}{\partial z} - e^{-2z} \frac{\partial^2\psi}{\partial z^2} + 2e^{-2z} \frac{\partial\psi}{\partial z})]\frac{\partial T}{\partial z}$$

$$- \lambda e^{-2z}[2 \frac{\partial G}{\partial\theta} - (\cot\theta)G - 2e^{-2z}(e^{2z}C^2)\psi] \frac{\partial T}{\partial\theta}$$

$$- \lambda e^{-2z}[2e^{-2z}(\frac{\partial^2\psi}{\partial z^2} - 2 \frac{\partial\psi}{\partial z}) - G](\frac{\partial^2 T}{\partial z^2} - \lambda(\frac{\partial T}{\partial z})^2 - \frac{\partial T}{\partial z} - \frac{\partial^2 T}{\partial\theta^2} + \lambda(\frac{\partial T}{\partial\theta})^2)$$

Equation (2.25) continued

$$- \lambda e^{-2z}[2e^{-2z}(e^{2z}C^2)\psi](\frac{\partial^2 T}{\partial z \partial\theta} - \lambda \frac{\partial T}{\partial z}\frac{\partial T}{\partial\theta}). \qquad (2.25)$$

(b) Dimensionless variable properties energy equation

$$\frac{N_{Re}}{2} N_{Pr}\{(1-\beta T)(1-\varepsilon T)[\frac{\partial T}{\partial t} - \frac{e^{-3z}}{\sin\theta}\frac{\partial(T,\psi)}{\partial(z,\theta)}]\} = e^{-2z}[(1-\gamma T) \qquad (2.26)$$

$$(\frac{\partial^2 T}{\partial z^2} + \frac{\partial T}{\partial z} + \frac{\partial^2 T}{\partial\theta^2} + \cot\theta\,\frac{\partial T}{\partial\theta}) - \gamma((\frac{\partial T}{\partial z})^2 + (\frac{\partial T}{\partial\theta})^2)].$$

2.3. Establishment of Boundary Conditions

The development of the generalized boundary conditions must be divided into two categories, namely the fluid-fluid system and the solid-fluid system. For a fluid sphere moving relative to another extensive fluid medium, the set of boundary conditions must include

(a) Boundary conditions at the sphere center (origin),

(b) Interfacial boundary conditions,

(c) Infinite extent boundary conditions, and

(d) Axes of symmetry boundary conditions.

For a solid sphere in motion relative to an extensive fluid medium the boundary conditions (c) and (d) above need to be specified, in addition to the conditions at the solid wall as well as within the solid sphere itself. In each case, the boundary conditions must be developed both from the viewpoint of accepted nomenclature and convention, with particular reference to their implication upon the nomenclature and symbols immediate to this work. These boundary conditions apply most specifically to the properties of temperature, pressure, and surface tension and to the flow parameters of stream function, vorticity, radial velocity component, tangential velocity component, and shear stress. Furthermore, these boundary conditions must consider either sedimentary or permeability flow extremes, as well as flow regimes which are a mixture of these two.

A complete listing of the applicable boundary conditions for fluid-fluid systems is provided in Table 2. A corresponding listing for solid-fluid systems may be found in Table 3. The permeability flow regime refers to the motion of a surrounding fluid medium relative to a stationary suspended body. The conventional streamline pattern for permeability flow is illustrated in Figure 3. The sedimentary flow regime refers to the motion of the suspended body moving relative to its surrounding stationary fluid medium. The conventional streamline pattern for the sedimentary flow regime is illustrated in Figure 4. Initial conditions are listed in Table 4. A few examples to illustrate the transition from the conventional dimensional form of the boundary conditions to the corresponding dimensionless form of the boundary conditions may be helpful. Three of the relatively complex conditions are chosen as examples for

Table 2. Itemized Boundary Conditions for Fluid-Fluid Systems

Conventional Dimensional Form $[w(r,\theta,t)]$	Corresponding Dimensionless Form $[w(z,\theta,t)]$	Rationale
$T^*(0,\theta,t)$ must be finite	$T^*(-z_\infty,\theta,t)$ must be finite	Finiteness at origin
$\psi^*(0,\theta,t)$ must be finite	$\psi^*(-z_\infty,\theta,t)$ must be finite	Finiteness at origin
$\zeta^*(0,\theta,t)$ must be finite	$\zeta^*(-z_\infty,\theta,t)$ must be finite	Finiteness at origin
$v_r^*(0,\theta,t)$ must be finite	$\frac{\partial\psi^*}{\partial\theta}(-z_\infty,\theta,t)$ must be finite	Finiteness at origin
$v_\theta^*(0,\theta,t)$ must be finite	$\frac{\partial\psi^*}{\partial z}(-z_\infty,\theta,t)$ must be finite	Finiteness at origin
$p^*(0,\theta,t)$ must be finite	$p^*(-z_\infty,\theta,t)$ must be finite	Finiteness at origin
$k^* \frac{\partial T^*}{\partial r}(0,\theta,t) = 0$	$\frac{\partial T^*}{\partial z}(-z_\infty,\theta,t) = 0$	Finiteness at origin
$\frac{k^*}{r}\frac{\partial T^*}{\partial\theta}(0,\theta,t) = 0$	$\frac{\partial T^*}{\partial\theta}(-z_\infty,\theta,t) = 0$	Finiteness at origin
$v_r^*(r_o,\theta,t) = -v_r(r_o,\theta,t) = \frac{dr_o(t)}{dt}$	$\frac{\partial\psi^*}{\partial\theta} = -\frac{\partial\psi}{\partial\theta} = -\frac{\beta\sin\theta}{3U}(1-\beta T)^{-\frac{4}{3}}$ $r_o(0)\frac{\partial T}{\partial t}$ at $(0,\theta,t)$	Impenetrable interface
$v_\theta^*(r_o,\theta,t) = v_\theta(r_o,\theta,t)$	$\frac{\partial\psi^*}{\partial z}(0,\theta,t) = \frac{\partial\psi}{\partial z}(0,\theta,t)$	No-slip interface

Table 2 (continued)

Conventional Dimensional Form $[w(r,\theta,t)]$	Corresponding Dimensionless Form $[w(z,\theta,t)]$	Rationale
$T^*(r_o,\theta,t) = T(r_o,\theta,t)$	$T^*(0,\theta,t) = T(0,\theta,t)$	No-jump temperature
$k^* \frac{\partial T^*}{\partial r} (r_o,\theta,t) = k \frac{\partial T}{\partial r} (r_o,\theta,t)$	$k^* \frac{\partial T^*}{\partial z} (0,\theta,t) = k \frac{\partial T}{\partial z} (0,\theta,t)$	Continuity of heat flux vector
$\frac{k^*}{r} \frac{\partial T^*}{\partial \theta} (r_o,\theta,t) = \frac{k}{r} \frac{\partial T}{\partial \theta} (r_o,\theta,t)$	$k^* \frac{\partial T^*}{\partial \theta} (0,\theta,t) = k \frac{\partial T}{\partial \theta} (0,\theta,t)$	Continuity of heat flux vector
$\psi^*(r,0,t) = \psi(r,0,t) = 0$	$\psi^*(z,0,t) = \psi(z,0,t) = 0$	Axially symmetric flow fields
$\zeta^*(r,0,t) = \zeta(r,0,t) = 0$	$\zeta^*(z,0,t) = \zeta(z,0,t) = 0$	Axially symmetric flow fields
$\psi^*(r,\pi,t) = \psi(r,\pi,t) = 0$	$\psi^*(z,\pi,t) = \psi(z,\pi,t) = 0$	Axially symmetric flow fields
$\zeta^*(r,\pi,t) = \zeta(r,\pi,t) = 0$	$\zeta^*(z,\pi,t) = \zeta(z,\pi,t) = 0$	Axially symmetric flow fields

Table 2 (continued)

Conventional Dimensional Form $[w(r,\theta,t)]$	Corresponding Dimensionless Form $[w(z,\theta,t)]$	Rationale
$(p-p^*)+(\tau_{rr}-\tau^*_{rr}) = -\frac{2\sigma}{r_o}$ at (r_o,θ,t)	$[p + \frac{e^{-\lambda T}}{\sin\theta}(\frac{\partial^2\psi}{\partial z\partial\theta} - 2\frac{\partial\psi}{\partial\theta})] - [p^* - \frac{e^{-\lambda^* T^*}}{\sin\theta}(\frac{\partial^2\psi^*}{\partial z\partial\theta} + 2\frac{\partial\psi^*}{\partial\theta})] = -\sigma$	Equality of stress at interface
$(p-p^*)+(\tau_{\theta\theta}-\tau^*_{\theta\theta}) = -\frac{2\sigma}{r_o}$ at (r_o,θ,t)	$-\mu(e^{2z}C^2)\psi = -\mu^*(e^{-2z}C^2)\psi^*$	Equality of stress at interface
$(p-p^*)+(\tau_{\phi\phi}-\tau^*_{\phi\phi}) = -\frac{2\sigma}{r_o}$ at (r_o,θ,t)	1) $\mu(\frac{\partial^2\psi}{\partial z\partial\theta} - 2\cot\theta\,\frac{\partial\psi}{\partial z}) = \mu^*(-\frac{\partial^2\psi^*}{\partial z\partial\theta} + 2\cot\theta\frac{\partial\psi^*}{\partial z})$	Equality of stress at interface
$(\tau_{r\theta}-\tau^*_{r\theta}) = -\frac{1}{r_o}\frac{\partial\sigma}{\partial\theta}$ at (r_o,θ,t)	2) $[-\frac{e^{-\lambda T}}{\sin\theta}(\frac{\partial^2\psi}{\partial z\partial\theta} - \cot\theta\,\frac{\partial\psi}{\partial z} - \frac{\partial\psi}{\partial\theta})] - [\frac{e^{-\lambda^* T^*}}{\sin\theta}(\frac{\partial^2\psi^*}{\partial z\partial\theta} - \cot\theta\,\frac{\partial\psi^*}{\partial z} + \frac{\partial\psi^*}{\partial\theta})] = -\frac{1}{2}\frac{\partial\sigma}{\partial\theta}$	Equality of stress at interface

above are applied at $(0,\theta,t)$

Table 2 (concluded)

Conventional Dimensional Form $[w(r,\theta,t)]$	Corresponding Dimensionless Form $[w(z,\theta,t)]$	Rationale
$T(\infty,\theta,t) = T_\infty$	$T(z_\infty,\theta,t) = 1$	Infinite extent in continuous medium
$\psi(\infty,\theta,t) = 0$	$\psi(z_\infty,\theta,t) = 0$	Infinite extent in continuous medium
$\zeta(\infty,\theta,t) = 0$	$\zeta(z_\infty,\theta,t) = 0$	Infinite extent in continuous medium
$k \frac{\partial T}{\partial r} (\infty,\theta,t) = 0$	$\frac{\partial T}{\partial z} (z_\infty,\theta,t) = 0$	Infinite extent in continuous medium
$\frac{k}{r} \frac{\partial T}{\partial \theta} (\infty,\theta,t) = 0$	$\frac{\partial T}{\partial \theta} (z_\infty,\theta,t) = 0$	Infinite extent in continuous medium

(1) By first subtracting τ_{rr} equation before transformation.

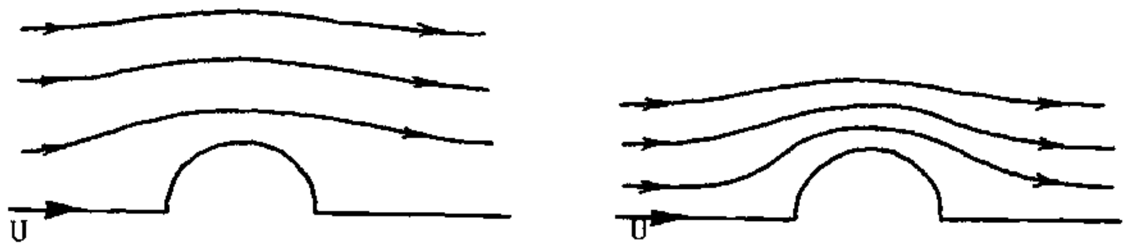
(2) By first subtracting $\tau_{\theta\theta}$ equation before transformation.

Table 3. Itemized Boundary Conditions for Solid-Fluid Systems

Conventional Dimensional Form $[w(r,\theta,t)]$	Corresponding Dimensionless Form $[w(z,\theta,t)]$	Rationale
$v_r(r_o,\theta,t) = 0$	$\frac{\partial\psi}{\partial z}(0,\theta,t) = 0$	Impenetrable interface
$v_\theta(r_o,\theta,t) = 0$	$\frac{\partial\psi}{\partial z}(0,\theta,t) = 0$	No-slip interface
$T^*(r_o,\theta,t) = T(r_o,\theta,t)$	$T^*(0,\theta,t) = T(0,\theta,t)$	No-jump temperature
$k^* \frac{\partial T^*}{\partial r}(r_o,\theta,t) = k \frac{\partial T}{\partial r}(r_o,\theta,t)$	$k^* \frac{\partial T^*}{\partial z}(0,\theta,t) = k \frac{\partial T}{\partial z}(0,\theta,t)$	Continuity of heat flux vector
$\frac{k^*}{r} \frac{\partial T^*}{\partial \theta}(r_o,\theta,t) = \frac{k}{r} \frac{\partial T}{\partial \theta}(r_o,\theta,t)$	$k^* \frac{\partial T^*}{\partial \theta}(0,\theta,t) = k \frac{\partial T}{\partial \theta}(0,\theta,t)$	Continuity of heat flux vector
$\psi^*(r,\theta,t) = \zeta^*(r,\theta,t) = 0$	$\psi^*(-z,\theta,t) = \zeta^*(-z,\theta,t) = 0$	Inside solid sphere
$p^*(r,\theta,t) = 0$	$p^*(-z,\theta,t) = 0$	Inside solid sphere

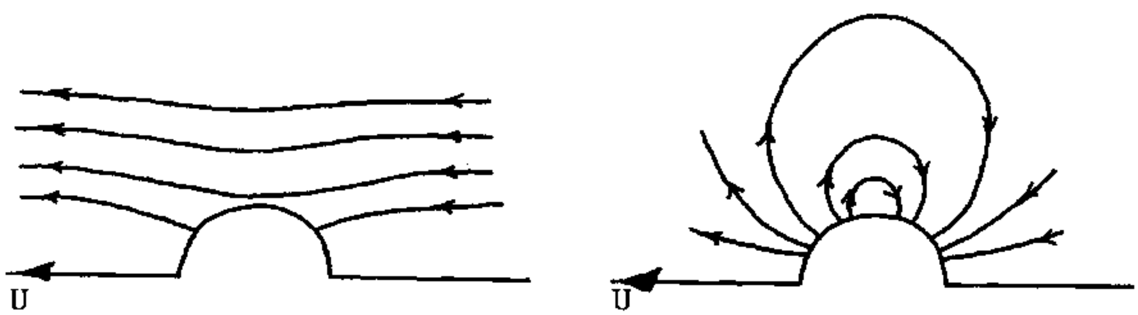
Table 3 (concluded)

Conventional Dimensional Form [$w(r,\theta,t)$	Corresponding Dimensionless Form [$w(z,\theta,t)$]	Rationale
$\psi(r,0,t) = 0$	$\psi(z,0,t) = 0$	Axially symmetric flow field
$\zeta(r,0,t) = 0$	$\zeta(z,0,t) = 0$	Axially symmetric flow field
$\psi(r,\pi,t) = 0$	$\psi(z,\pi,t) = 0$	Axially symmetric flow field
$\zeta(r,\pi,t) = 0$	$\zeta(z,\pi,t) = 0$	Axially symmetric flow field
$T(\infty,\theta,t) = T_\infty$	$T(z_\infty,\theta,t) = 1$	Infinite extent in continuous medium
$\psi(\infty,\theta,t) = 0$	$\psi(z_\infty,\theta,t) = 0$	Infinite extent in continuous medium
$\zeta(\infty,\theta,t) = 0$	$\zeta(z_\infty,\theta,t) = 0$	Infinite extent in continuous medium
$k\,\frac{\partial T}{\partial r}(\infty,\theta,t) = 0$	$\frac{\partial T}{\partial z}(z_\infty,\theta,t) = 0$	Infinite extent in continuous medium
$\frac{k}{r}\,\frac{\partial T}{\partial \theta}(\infty,\theta,t) = 0$	$\frac{\partial T}{\partial \theta}(z_\infty,\theta,t) = 0$	Infinite extent in continuous medium



a. Stokes flow
b. Potential flow

Figure 3. Permeability Flow Regime Streamlines (White, 1974)



a. Stokes flow
b. Potential flow

Figure 4. Sedimentary Flow Regime Streamlines (White, 1974)

Table 4. Initial Conditions for Fluid-Fluid and Solid-Fluid Systems

$T(r,\theta,0) = T_\infty$	$T(z,\theta,0) = 1$
$T^*(r,\theta,0) = T_o$	$T^*(z,\theta,0) = 0$
$\zeta^*(a,\theta,0) = E^2\psi^*(a,\theta,0)$	$\zeta^*(0,\theta,0) = e^{2z}[(e^{-2z}E^2)\psi(0,\theta,0)]$
$\zeta(a,\theta,0) = E^2\psi(a,\theta,0)$	$\zeta(0,\theta,0) = e^{-2z}[(e^{2z}E^2)\psi(0,\theta,0)]$
$\psi^*(r,\theta,0) = \psi(r,\theta,0) = 0$	$\psi^*(z,\theta,0) = \psi(z,\theta,0) = 0$

this purpose. First, the no-penetration interfacial boundary condition for fluid-fluid systems is developed into its dimensionless form.

It is assumed that although the temperature, and hence density, of the inner fluid will change through the course of the experiment, the initial fluid mass

$$m = \rho(0)\forall(0) = \rho(0)\ \frac{4}{3}\ \pi(r_o(0))^3 = \rho(t)\ \frac{4}{3}\ \pi(r_o(t))^3, \quad (2.27)$$

will remain constant, where $r_o(t)$ expresses the interfacial radius at time t. Equation (2.27), on differentiation, gives

$$\frac{dr_o}{dt} = -\frac{r_o(0)}{3}\sqrt[3]{\frac{\rho(0)}{\rho(t)}}\ \frac{1}{\rho(t)}\ \frac{d\rho(t)}{dt}\ . \quad (2.28)$$

On substitution of the Boussinesq relationship of Equation (2.21) into Equation (2.28) gives

$$\frac{dr_o}{dt} = \frac{\beta}{3}\ r_o(0)(1-\beta T)^{-\frac{4}{3}}\ \frac{dT}{dt}\ . \quad (2.29)$$

Since, at the interface, the radial velocity components for the internal and external fluids must be equal in magnitude and opposed in sense, a boundary condition of the form

$$v_r^*(r_o,\theta,t) = -v_r(r_o,\theta,t) = \frac{dr_o}{dt}, \tag{2.30}$$

may be expressed as

$$-\frac{U}{\sin\theta}\frac{\partial\psi^*}{\partial\theta}(r_o,\theta,t) = \frac{U}{\sin\theta}\frac{\partial\psi}{\partial\theta}(r_o,\theta,t) \tag{2.31}$$

$$= r_o'(t) = \frac{\beta}{3} r_o(0)(1-\beta T)^{-\frac{4}{3}}\frac{dT}{dt}.$$

The corresponding dimensionless form of boundary condition becomes

$$\frac{\partial\psi^*}{\partial\theta}(0,\theta,t) = -\frac{\partial\psi}{\partial\theta}(0,\theta,t) \tag{2.32}$$

$$= -\frac{\beta}{3} r_o(0)(1-\beta T)^{-\frac{4}{3}}\frac{dT}{dt}(0,\theta,t).$$

Next, the continuity of stress demands an equality of the stress tensor at the interface. The complete stress tensor equality includes shear stresses and normal stresses, including surface tension components, and may be stated as

$$\tau - \tau^* = \sigma \quad \text{at} \quad r = r_o, \text{ or} \tag{2.33}$$

$$\begin{bmatrix} p+\tau_{rr} & \tau_{\theta r} & 0 \\ \tau_{r\theta} & p+\tau_{\theta\theta} & 0 \\ 0 & 0 & p+\tau_{\phi\phi} \end{bmatrix} - \begin{bmatrix} p^*+\tau^*_{rr} & \tau^*_{\theta r} & 0 \\ \tau^*_{r\theta} & p^*+\tau^*_{\theta\theta} & 0 \\ 0 & 0 & p^*+\tau^*_{\phi\phi} \end{bmatrix} = \tag{2.34}$$

$$\begin{bmatrix} -\frac{2\sigma}{r_o} & -\frac{1}{r_o}\frac{d\sigma}{d\theta} & 0 \\ -\frac{1}{r_o}\frac{d\sigma}{d\theta} & -\frac{2\sigma}{r_o} & 0 \\ 0 & 0 & -\frac{2\sigma}{r_o} \end{bmatrix} \quad \text{at } r = r_o.$$

For example, the rr component equality, on substitution of the viscous stresses in terms of velocity gradients, (see Appendix A), gives

$$(p-2\mu\,\frac{\partial v_r}{\partial r}) - (p^*-2\mu^*\,\frac{\partial v_r^*}{\partial r}) = -\,\frac{2\sigma}{r_o}\,. \tag{2.35}$$

Nondimensionalization and spatial transformation ($r = e^z$) of Equation (2.35) gives

$$[p + \frac{e^{-\lambda T}}{\sin\theta}\,(\frac{\partial^2\psi}{\partial z\partial\theta} - 2\,\frac{\partial\psi}{\partial\theta})] - [p^* - \frac{e^{-\lambda^* T^*}}{\sin\theta}\,(\frac{\partial^2\psi^*}{\partial z\partial\theta} + 2\,\frac{\partial\psi^*}{\partial\theta})] = -\sigma \tag{2.36}$$
$$\text{at } (\phi,\theta,t).$$

A similar examination of the $r\theta$ component equality gives

$$\left(-2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r}{r}\right)\right)-\left(-2\mu^*\left(\frac{1}{r}\frac{\partial v_\theta^*}{\partial \theta}+\frac{v_r^*}{r}\right)\right) = -\frac{1}{r_o}\frac{d\sigma}{d\theta} \qquad (2.37)$$

$$\text{at } (r_o,\theta,t),$$

which may be nondimensionalized and spatially transformed to yield

$$\left[-\frac{e^{-\lambda T}}{\sin\theta}\left(\frac{\partial^2\psi}{\partial z\partial\theta}-\cot\theta\,\frac{\partial\psi}{\partial\theta}-\frac{\partial\psi}{\partial\theta}\right)\right]-\left[\frac{e^{-\lambda^* T^*}}{\sin\theta}\left(\frac{\partial^2\psi^*}{\partial z\partial\theta}-\cot\theta\,\frac{\partial\psi^*}{\partial z}+\frac{\partial\psi^*}{\partial\theta}\right)\right] = -\frac{1}{2}\frac{d\sigma}{d\theta} \text{ at } (0,\theta,t). \qquad (2.38)$$

The foregoing discussion on the development of boundary conditions leading to equations (2.32), (2.36), and (2.38) indicate the extent of complexity involved in detailing the process. A similar analysis pertaining to the remaining boundary and initial conditions leads to the concise listings displayed via Tables 2, 3, 4, and 5.

In summary, the dynamics and thermal equilibrium of a disperse droplet in relative motion with respect to a continuous medium must be examined via Equations (2.23), (2.24), (2.25), and (2.26), subject to the applicable boundary and initial conditions given in Tables 2, 3, 4, and 5.

2.4. Statement of the Specific Problem

In order to establish the feasibility of the mathematical model a specific problem that is explored in detail is that of the initial isothermal developing flow around a solid sphere in motion in an intermediate Reynolds number range ($1 < Re < 40$) through an incompressible Newtonian fluid of infinite extent. Additional assumptions needed to be imposed upon the general model developed in the preceding analysis are as follows

(i) Constant temperature exists at all locations of the internal (solid) and external (fluid) fields,

(ii) Fluid properties are constant through the internal and external field, and

(iii) Potential flow solution exists on the solid surface initially.

As a consequence of the additional assumptions imposed for the general problem, the energy equation does not need to be examined, yielding the principle equation of transport as the vorticity-stream function equation. In order to recognize the full significance of these assumptions it is necessary to express the dimensionless constant properties vorticity-stream function equation for the external fluid, more simply as

$$\frac{N_{Re}}{2}\left[e^{2z}\frac{\partial E^2\psi}{\partial t} - \frac{\partial\left(\frac{E^2\psi}{e^{2z}\sin^2\theta},\psi\right)}{\partial(z,\theta)}e^{z}\sin\theta\right] = (e^{2z}E^2)(E^2\psi). \quad (2.39)$$

CHAPTER III

COMPUTATIONAL FORMULATION

3.1. Computational Formulation for the General Problem

In an effort to facilitate numerical computation of the flow field within and about the droplet, the interrelated works of finite-differencing of the major governing equations and their boundary and initial conditions are broken down into the following groups:

(a) Normal spatial operators, such as $\frac{\partial}{\partial\theta}$, $\frac{\partial}{\partial z}$, $\frac{\partial^2}{\partial\theta^2}$, and $\frac{\partial^2}{\partial z^2}$,

(b) Boundary spatial operators, such as $(\frac{\partial}{\partial\theta})_{0,\theta}$, $(\frac{\partial}{\partial z})_{0,\theta}$, ...,

(c) Jacobian partial differential operator, $\frac{\partial(f,g)}{\partial(z,\theta)}$,

(d) Second differential in two directions, $\frac{\partial^2}{\partial z\partial\theta}$, and

(e) Temperature equation.

Figure 5 represents a diagrammatic flow chart of the crucial steps in the overall solution synthesis using computational techniques.

In developing the finite difference forms of the partial differential operations, consider the Taylor-series expansions about the point (i,j) denoted as a mesh point. Its location with respect to the complete mesh system is

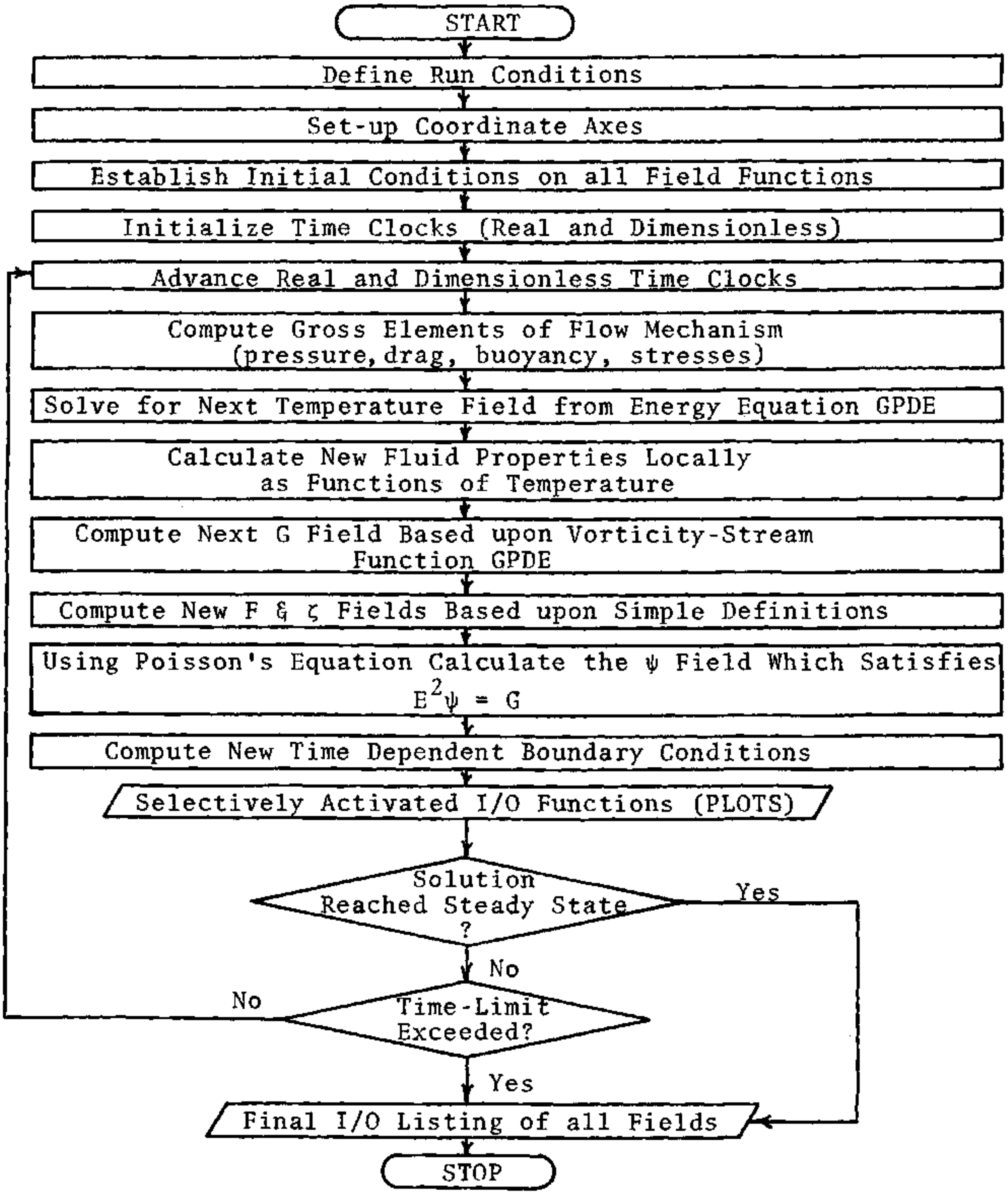


Figure 5. Diagrammatic Flow Chart of Critical Path of Overall Problem Synthesis

illustrated for the internal fluid in Figure 6, and for the external fluid in Figure 7. These expansions, for the internal fluid, for a function $w(z,\theta)$ are

$$w_{i,j+1} = w_{i,j} + b\,\frac{\partial w}{\partial \theta} + \frac{b^2}{2}\,\frac{\partial^2 w}{\partial \theta^2}\,, \tag{3.1}$$

$$w_{i+1,j} = w_{i,j} - a\,\frac{\partial w}{\partial z} + \frac{a^2}{2}\,\frac{\partial^2 w}{\partial z^2}\,,$$

$$w_{i,j-1} = w_{i,j} - b\,\frac{\partial w}{\partial \theta} + \frac{b^2}{2}\,\frac{\partial^2 w}{\partial \theta^2}\,,$$

and $$w_{i-1,j} = w_{i,j} + a\,\frac{\partial w}{\partial z} + \frac{a^2}{2}\,\frac{\partial^2 w}{\partial z^2}\,,$$

while for the external fluid, they are

$$w_{i,j+1} = w_{i,j} + b\,\frac{\partial w}{\partial \theta} + \frac{b^2}{2}\,\frac{\partial^2 w}{\partial \theta^2}\,, \tag{3.2}$$

$$w_{i+1,j} = w_{i,j} + a\,\frac{\partial w}{\partial z} + \frac{a^2}{2}\,\frac{\partial^2 w}{\partial z^2}\,,$$

$$w_{i,j-1} = w_{i,j} - b\,\frac{\partial w}{\partial \theta} + \frac{b^2}{2}\,\frac{\partial^2 w}{\partial \theta^2}\,,$$

and $$w_{i-1,j} = w_{i,j} - a\,\frac{\partial w}{\partial z} + \frac{a^2}{2}\,\frac{\partial^2 w}{\partial z^2}\,.$$

These systems of equations, accurate to the second order, may be solved to yield the finite difference forms of the elementary partial differential operators. Those operators are, for the internal fluid,

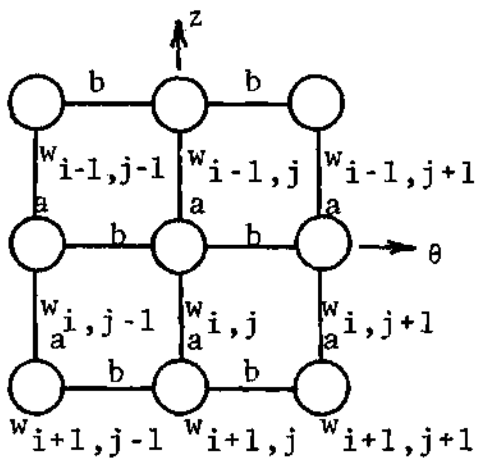


Figure 6. Diagram of Internal Mesh Layout

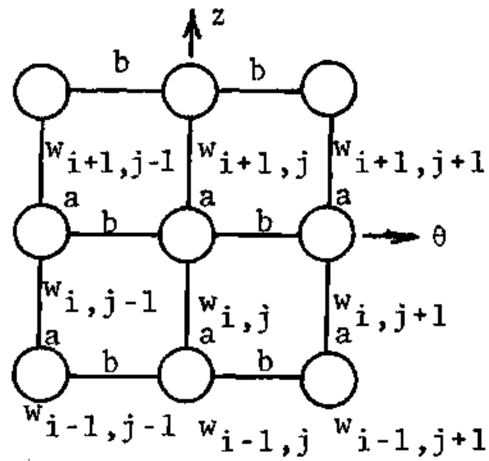


Figure 7. Diagram of External Mesh Layout

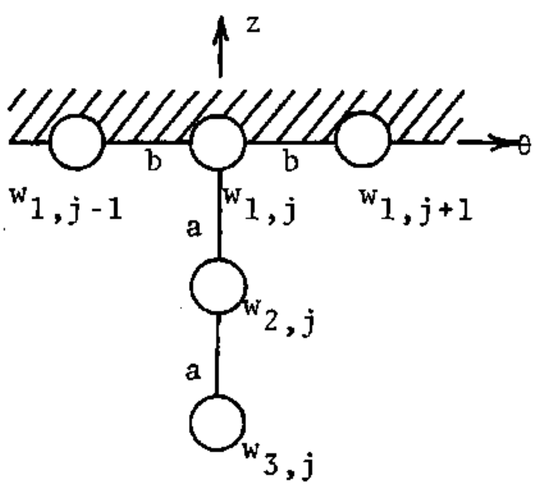


Figure 8. Diagram of Internal Interface Mesh Layout

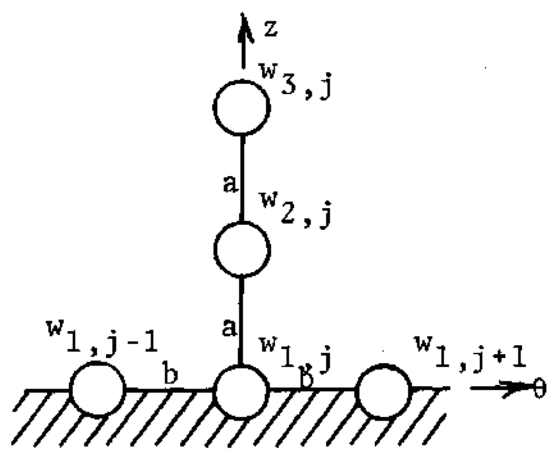


Figure 9. Diagram of External Interface Mesh Layout

$$\frac{\partial w}{\partial z} = -\frac{w_{i+1,j} - w_{i-1,j}}{2a}, \tag{3.3}$$

$$\frac{\partial w}{\partial \theta} = \frac{w_{i,j+1} - w_{i,j-1}}{2b},$$

$$\frac{\partial^2 w}{\partial z^2} = \frac{w_{i+1,j} + w_{i-1,j}}{a^2} - \frac{2}{a^2} w_{i,j},$$

and

$$\frac{\partial^2 w}{\partial \theta^2} = \frac{w_{i,j+1} + w_{i,j-1}}{b^2} - \frac{2}{b^2} w_{i,j},$$

while for the external fluid, these are

$$\frac{\partial w}{\partial z} = \frac{w_{i+1,j} - w_{i-1,j}}{2a}, \tag{3.4}$$

$$\frac{\partial w}{\partial \theta} = \frac{w_{i,j+1} - w_{i,j-1}}{2b},$$

$$\frac{\partial^2 w}{\partial z^2} = \frac{w_{i+1,j} + w_{i-1,j}}{a^2} - \frac{2}{a^2} w_{i,j},$$

and

$$\frac{\partial^2 w}{\partial \theta^2} = \frac{w_{i,j+1} + w_{i,j-1}}{b^2} - \frac{2}{b^2} w_{i,j}.$$

By direct application of the preceding the velocity forms become, for the internal fluid

$$v_{r_{i,j}} = -\frac{U}{e^{-2z}\sin\theta}\frac{\partial\psi}{\partial\theta} = -\frac{e^{2z_i}}{\sin\theta_j}\frac{U}{2b}(\psi_{i,j+1} - \psi_{i,j-1}), \tag{3.5}$$

Equation (3.5) continued

and $$v_{\theta_{i,j}} = - \frac{U}{e^{-2z}\sin\theta} \frac{\partial\psi}{\partial z} = \frac{e^{2z_i}}{\sin\theta_j} \frac{U}{2a} (\psi_{i+1,j} - \psi_{i-1,j}), \quad (3.5)$$

while for the external fluid these yield

$$v_{r_{i,j}} = - \frac{1}{e^{2z}\sin\theta} \frac{\partial\psi}{\partial\theta} = - \frac{1}{e^{2z_i} \sin\theta_j} \frac{U}{2b} (\psi_{i,j+1} - \psi_{i,j-1}), \quad (3.6)$$

and $$v_{\theta_{i,j}} = \frac{1}{e^{2z}\sin\theta} \frac{\partial\psi}{\partial z} = \frac{1}{e^{2z_i} \sin\theta_j} \frac{U}{2a} (\psi_{i+1,j} - \psi_{i-1,j}).$$

Furthermore, the finite difference forms for the complex partial differential operator, $e^{-2z}E^2\psi_{i,j}$, for the internal flow, and the corresponding operator, $e^{2z}E^2\psi_{i,j}$, for the external fluid, are identical. These are given as

$$(e^{-2z}E^2)(\psi_{i,j}) = (e^{2z}E^2)(\psi_{i,j}) = \quad (3.7)$$

$$\left(\frac{2-a}{2a^2} \psi_{i+1,j} + \frac{2-b\cot\theta_j}{2b^2} \psi_{i,j+1} + \frac{2+a}{2a^2} \psi_{i-1,j} + \frac{2+b\cot\theta_j}{2b^2} \psi_{i,j-1}\right)$$

$$- \left(\frac{2}{a^2} + \frac{2}{b^2}\right)\psi_{i,j}.$$

The development of the finite difference forms of the partial differential operations involving boundary nodes may be approached in a similar fashion. For instance, consider the Taylor-series expansions about the point (1,j) along the interfacial boundary, described in Figures 8 and 9, and denoted as a mesh point. These expansions for the internal fluid are

$$w_{1,j+1} = w_{1,j} + b \frac{\partial w}{\partial \theta} + \frac{b^2}{2} \frac{\partial^2 w}{\partial \theta^2} , \tag{3.8}$$

$$w_{2,j} = w_{1,j} - a \frac{\partial w}{\partial z} + \frac{a^2}{2} \frac{\partial^2 w}{\partial z^2} ,$$

$$w_{3,j} = w_{1,j} - 2a \frac{\partial w}{\partial z} + 4 \frac{a^2}{2} \frac{\partial^2 w}{\partial z^2} ,$$

and

$$w_{1,j-1} = w_{1,j} - b \frac{\partial w}{\partial \theta} + \frac{b^2}{2} \frac{\partial^2 w}{\partial \theta^2} ,$$

while for the external fluid, they are

$$w_{1,j+1} = w_{1,j} + b \frac{\partial w}{\partial \theta} + \frac{b^2}{2} \frac{\partial^2 w}{\partial \theta^2} , \tag{3.9}$$

$$w_{2,j} = w_{1,j} + a \frac{\partial w}{\partial z} + \frac{a^2}{2} \frac{\partial^2 w}{\partial z^2} ,$$

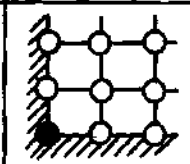
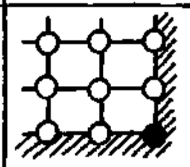
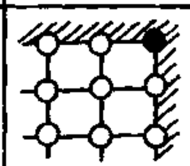
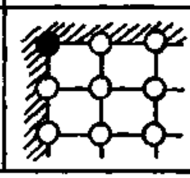
$$w_{3,j} = w_{1,j} + 2a \frac{\partial w}{\partial z} + 4 \frac{a^2}{2} \frac{\partial^2 w}{\partial z^2} ,$$

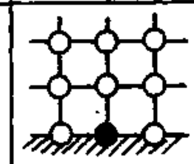
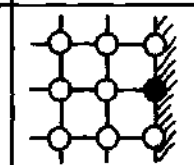
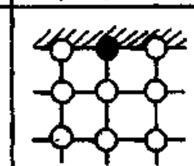
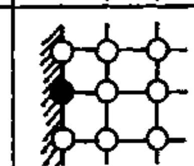
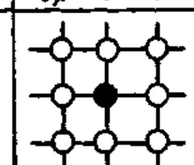
and

$$w_{1,j-1} = w_{1,j} - b \frac{\partial w}{\partial \theta} + \frac{b^2}{2} \frac{\partial^2 w}{\partial \theta^2} .$$

These systems of equations, also accurate to the second order, may be solved to yield the finite difference forms of the elementary partial differential operators. These operators are, for the internal fluid,

Table 5. Two Directional Partial Differential Operator $\frac{\partial^2}{\partial z \partial \theta}$ in Finite Difference Form

Physical Description	Indicial Location (External / Internal)	Schematic Location	Stencil (Multipliers)
Corner Boundary Forms	(1,1) (m+1,1)		3 -4 1 -12 16 -4 9 -12 3
	(1,n+1) (m+1,n+1)		-1 4 -3 4 -16 12 -3 12 -9
	(m+1,n+1) (1,n+1)		3 -12 9 -4 16 -12 1 -4 3
	(m+1,1) (1,1)		-9 12 -3 12 -16 4 -3 4 -1

Physical Description	Indicial Location (External / Internal)	Schematic Location	Stencil (Multipliers)
Edge Boundary Forms	(1,j) (m+1,j)		1 - -1 -4 - 4 3 - -3
	(i,n+1) (i,n+1)		1 -4 3 - - - -1 4 -3
	(m+1,j) (1,j)		-3 - 3 4 - -4 -1 - 1
	(i,1) (i,1)		-3 4 -1 - - - 3 -4 1
Non-Boundary Form	(i,j) (i,j)		-1 - 1 - - - 1 - -1

Key to Schematic

- Edge Node
- Corner node
- Spatial node
- Location of node central to operation

Use: Multiply corresponding neighborhood values of field array by appropriate multipliers, sum the products, and divide by (4ab). For example, the value of $\frac{\partial^2}{\partial z \partial \theta}$ in the corner (i,j) for the external fluid is $\frac{\partial^2}{\partial z \partial \theta}(w_{i,j}) =$

$$\frac{-w_{i+1,j-1}+w_{i+1,j+1}-w_{i-1,j+1}+w_{i-1,j-1}}{4ab}.$$

$$\frac{\partial w}{\partial z} = \frac{3w_{1,j} - 4w_{2,j} + w_{3,j}}{2a} , \tag{3.10}$$

$$\frac{\partial w}{\partial \theta} = \frac{w_{1,j+1} - w_{1,j-1}}{2b} ,$$

$$\frac{\partial^2 w}{\partial z^2} = \frac{w_{1,j} - 2w_{2,j} + w_{3,j}}{a^2} ,$$

and

$$\frac{\partial^2 w}{\partial \theta^2} = \frac{w_{1,j+1} + w_{1,j-1}}{b^2} - \frac{2}{b^2} w_{1,j},$$

while for the external fluid,

$$\frac{\partial w}{\partial z} = \frac{-3w_{1,j} + 4w_{2,j} - w_{3,j}}{2a} , \tag{3.11}$$

$$\frac{\partial w}{\partial \theta} = \frac{w_{1,j+1} - w_{1,j-1}}{2b} ,$$

$$\frac{\partial^2 w}{\partial z^2} = \frac{w_{1,j} - 2w_{2,j} + w_{3,j}}{a^2} ,$$

and

$$\frac{\partial^2 w}{\partial \theta^2} = \frac{w_{1,j+1} + w_{1,j-1}}{b^2} - \frac{2}{b^2} w_{1,j}.$$

By direct application of the preceding boundary forms the internal and external velocity forms at the interface become identical, namely,

$$v_{r_{1,j}} = -\frac{U}{e^{-2z}\sin\theta}\frac{\partial\psi}{\partial\theta} = -\frac{1}{\sin\theta_j}\frac{U}{2b}(\psi_{1,j+1}-\psi_{1,j-1}) = \tag{3.12}$$

$$-\frac{U}{e^{2z}\sin\theta}\frac{\partial\psi}{\partial\theta},$$

and $v_{\theta_{1,j}} = -\frac{U}{e^{-2z}\sin\theta}\frac{\partial\psi}{\partial z} = -\frac{1}{\sin\theta_j}\frac{U}{2a}(3\psi_{1,j}-4\psi_{2,j}+\psi_{3,j}) =$

$$-\frac{U}{e^{2z}\sin\theta}\frac{\partial\psi}{\partial z}.$$

Furthermore, the finite difference forms for both the internal and external fluids are identically the same for the complex partial differential operator, as indicated by Equation (3.7), thus,

$$(e^{-2z}E^2)(\psi_{1,j}) = (e^{2z}E^2)(\psi_{1,j}) = \tag{3.13}$$

$$\left(\frac{2+3a}{2a^2}\psi_{1,j} - \frac{2+2a}{a^2}\psi_{2,j} + \frac{2+a}{2a^2}\psi_{3,j}\right) +$$

$$\left(\frac{2+b\cot\theta_j}{2b^2}\psi_{1,j-1} - \frac{2}{b^2}\psi_{1,j} + \frac{2-b\cot\theta_j}{2b^2}\psi_{1,j+1}\right).$$

Another special case for consideration is the interfacial boundary conditions when applied to the rigid or solid sphere. For solid sphere problems, the radial and tangential velocities are zero on the interface boundary. For example, the third order accurate Taylor-series expansions for the external fluid only about the point (1,j) denoted as a mesh point are

$$w_{1,j+1} = w_{1,j} + \frac{b^2}{2}\frac{\partial^2 w}{\partial\theta^2} + \frac{b^3}{6}\frac{\partial^3 w}{\partial\theta^3}, \tag{3.14}$$

$$w_{2,j} = w_{1,j} + \frac{a^2}{2}\frac{\partial^2 w}{\partial z^2} + \frac{a^3}{6}\frac{\partial^3 w}{\partial z^3},$$

$$w_{3,j} = w_{1,j} + 4\frac{a^2}{2}\frac{\partial^2 w}{\partial z^2} + 8\frac{a^3}{6}\frac{\partial^3 w}{\partial z^3},$$

and

$$w_{1,j-1} = w_{1,j} + \frac{b^2}{2}\frac{\partial^2 w}{\partial\theta^2} - \frac{b^3}{6}\frac{\partial^3 w}{\partial\theta^3}.$$

This system of equations, accurate to the third order, may be solved to yield the finite difference forms of the elementary partial differential operators, namely,

$$\frac{\partial w}{\partial z} = 0, \tag{3.15}$$

$$\frac{\partial w}{\partial\theta} = 0,$$

$$\frac{\partial^2 w}{\partial z^2} = \frac{-7w_{1,j}+8w_{2,j}-w_{3,j}}{2a^2},$$

Equation (3.15) continued

and $$\frac{\partial^2 w}{\partial \theta^2} = \frac{w_{1,j+1}+w_{1,j-1}}{b^2} - \frac{2}{b^2} w_{1,j}. \tag{3.15}$$

Again, the preceding, by direct application, gives the external velocity forms on the interfacial boundary as

$$vr_{1,j} = 0 \tag{3.16}$$

$$v\theta_{1,j} = 0,$$

while the complex partial differential operator on the boundary becomes

$$(e^{2z}E^2)(\psi_{1,j}) = \left(\frac{-7\psi_{1,j}+8\psi_{2,j}-\psi_{3,j}}{2a^2}\right) + \tag{3.17}$$

$$\left(\frac{\psi_{1,j+1}+\psi_{1,j-1}}{b^2} - \frac{2}{b^2}\psi_{1,j}\right),$$

which, for the boundary condition, $\psi_{1,k} = 0$, becomes

$$(e^{2z}E^2)(\psi_{1,j}) = \frac{8\psi_{2,j}-\psi_{3,j}}{2a^2} . \tag{3.18}$$

The next item of interest is the development of the finite-difference form of the Jacobian transform. By process of straightforward substitution into the expression

$$\frac{\partial(f,g)}{\partial(z,\theta)} = \frac{\partial f}{\partial z}\frac{\partial g}{\partial \theta} - \frac{\partial f}{\partial \theta}\frac{\partial g}{\partial z}\,, \tag{3.19}$$

the finite difference form of the non-boundary Jacobian spatial operator for the internal fluid becomes

$$\frac{\partial(f,g)}{\partial(z,\theta)} = -\frac{1}{4ab}\,[(f_{i+1,j}-f_{i-1,j})(g_{i,j+1}-g_{i,j-1}) - \tag{3.20}$$

$$(f_{i,j+1}-f_{i,j-1})(g_{i+1,j}-g_{i-1,j})],$$

while for the external fluid, the Jacobian becomes

$$\frac{\partial(f,g)}{\partial(z,\theta)} = \frac{1}{4ab}\,[(f_{i+1,j}-f_{i-1,j})(g_{i,j+1}-g_{i,j-1}) - \tag{3.21}$$

$$(f_{i,j+1}-f_{i,j-1})(g_{i+1,j}-g_{i-1,j})].$$

A completely new second order spatial operator must be introduced into the variable properties vorticity-stream function equation in order to maintain its conciseness. Its existence is motivated by the spatial derivatives of the viscosity in Equations (2.23) and (2.25). In order to complete the finite-difference form of this new spatial operator, C^2, the second order differential in two directions must also be developed. Table 6 displays a complete list of the boundary and non-boundary forms of the second order differential in two directions. An example of the derivation is included for clarity. The development of the non-boundary form considers the series expansions of Equations (3.1) and (3.2) and the mesh configurations illustrated in Figures 4 and 5. By application of

$$\frac{\partial w}{\partial z} = - \frac{w_{i+1,j} - w_{i-1,j}}{2a} \text{, and} \tag{2.22}$$

$$\frac{\partial w}{\partial \theta} = \frac{w_{i,j+1} - w_{i,j-1}}{2b} ,$$

to each other simultaneously, the result is

$$\frac{\partial^2 w}{\partial z \partial \theta} = \frac{\partial}{\partial z} \left(\frac{\partial w}{\partial \theta}\right) = \frac{\partial}{\partial z} \left(- \frac{w_{i,j+1} - w_{i,j-1}}{2b}\right) = \tag{3.23}$$

$$- \frac{1}{2b} \frac{\partial}{\partial z} (w_{i,j+1}) + \frac{1}{2b} \frac{\partial}{\partial z} (w_{i,j-1}),$$

which expands into the complete expression for the internal fluid as

$$\frac{\partial^2 w}{\partial z \partial \theta} = - \frac{w_{i+1,j+1} - w_{i-1,j+1} + w_{i-1,j-1} - w_{i+1,j-1}}{4ab} . \tag{3.24}$$

Similarly, for the external fluid, the second partial becomes

$$\frac{\partial^2 w}{\partial z \partial \theta} = \frac{w_{i+1,j+1} - w_{i-1,j+1} + w_{i-1,j-1} - w_{i+1,j-1}}{4ab} . \qquad (3.25)$$

The development of the forms for the spatial operator C^2 is now within reach. By simple substitution, this operator for both internal and external fluids becomes identically the same expression. It is given as

$$(e^{-2z}C^2) = (e^{2z}C^2) = \frac{1}{2ab} (w_{i+1,j+1} - w_{i-1,j+1} + w_{i-1,j-1} - w_{i+1,j-1}) \qquad (3.26)$$

$$- \frac{\cot\theta_j}{2a} (w_{i+1,j} - w_{i-1,j}) - \frac{3}{2b} (w_{i,j+1} - w_{i,j-1}) .$$

The Laplacian spatial operator is central to the energy equation. By simple substitution, the conventional forms of the Laplacian for both the internal fluid

$$\nabla^2 w = \frac{\partial^2 w}{\partial z^2} - \frac{\partial w}{\partial z} + \frac{\partial^2 w}{\partial \theta^2} + \cot\theta_j \frac{\partial w}{\partial \theta} , \qquad (3.27)$$

as well as for the external fluid,

$$\nabla^2 w = \frac{\partial^2 w}{\partial z^2} + \frac{\partial w}{\partial z} + \frac{\partial^2 w}{\partial \theta^2} + \cot\theta_j \frac{\partial w}{\partial \theta} , \qquad (3.28)$$

are rendered into identical finite-difference forms for the non-boundary mesh points. Hence

$$\nabla^2 w = \left(\frac{2+a}{2a^2} w_{i+1,j} + \frac{2+b\ \cot\theta_j}{2b^2} w_{i,j+1} + \frac{2-a}{2a^2} w_{i-1,j} + \right. \qquad (3.29)$$

$$\left.\frac{2-b\ \cot\theta_j}{2b^2} w_{i,j-1}\right) - \left(\frac{2}{a^2} + \frac{2}{b^2}\right) w_{i,j}.$$

Finally, the energy equation for the new time-step predicts a new temperature field (superscript prime) for the internal fluid according to the relation

$$T'_{i,j} = T_{i,j} + \frac{c}{e^{3z_i} \sin\theta_j} \frac{\partial(T,\psi)}{\partial(z,\theta)} \qquad (3.30)$$

$$+ \frac{c}{e^{2z_i} \frac{N_{Re}}{2} N_{Pr}\ (1-\beta T)(1-\varepsilon T)} [(1-\gamma T)\nabla^2 T - \gamma((\frac{T_{i+1,j} - T_{i-1,j}}{2a})^2 +$$

$$(\frac{T_{i,j+1} - T_{i,j-1}}{2b})^2)],$$

whereas the new temperature field for the external fluid is predicted by the relation

$$T'_{i,j} = T_{i,j} + \frac{ce^{3z_i}}{\sin\theta_j} \frac{\partial(T,\psi)}{\partial(z,\theta)} \qquad (3.31)$$

$$+ \frac{ce^{2z_i}}{\frac{N_{Re}}{2} N_{Pr}(1-\beta T)(1-\varepsilon T)} [(1-\gamma T)\nabla^2 T - \gamma((- \frac{T_{i+1,j} - T_{i-1,j}}{2a})^2 +$$

Equation (3.31) continued

$$\left(\frac{T_{i,j+1}-T_{i,j-1}}{2b}\right)^2)]. \qquad (3.31)$$

3.2. Computational Formulation for the Specific Problem

The specific finite differencing scheme for both the boundary and the non-boundary regions developed in the preceding analysis may be applied to the specific problem of transient isothermal flow around a solid sphere at intermediate Reynolds numbers. The governing partial differential equation for this problem is the constant property vorticity-stream function equation, which is a fourth-order non-linear equation, and as such might not prove directly solvable. However, via the use of the vorticity function, ζ, steady state solutions to Equation (2.39) may be obtained. This technique produces two simultaneous second-order partial differential equations which are solvable, one of which is the Poisson's equation in a spherical frame. Equation (2.39) may thus be uncoupled into the spherical Poisson equation,

$$(e^{2z}E^2)(\psi) - (\zeta\ e^{3z}\sin\theta) = 0, \qquad (3.32)$$

and the constant properties vorticity-stream function equation

$$\frac{N_{Re}}{2}\ [e^{2z}\ \frac{\partial G}{\partial t} - \frac{\partial(F,\psi)}{\partial(z,\theta)}\ e^{z}\sin\theta] = (e^{2z}E^2)(G). \qquad (3.33)$$

The corresponding indicial form for the spherical Poisson equation is

$$(e^{2z}E^2)(\psi_{i,j}) = (\zeta_{i,j}e^{z_i}\sin\theta_j)(e^{2z_i}) = (e^{2z_i})(G_{i,j}), \tag{3.34}$$

while for the vorticity-stream function equation, the indicial form is

$$e^{2z_i}\frac{\partial G_{i,j}}{\partial t} + \left(\frac{\partial\psi_{i,j}}{\partial z}\frac{\partial F_{i,j}}{\partial\theta} - \frac{\partial F_{i,j}}{\partial z}\frac{\partial\psi_{i,j}}{\partial\theta}\right)e^{z_i}\sin\theta_j = \tag{3.35}$$

$$\frac{2}{N_{Re}}(e^{2z}E^2)(G_{i,j}).$$

The spherical Poisson equation in finite difference form becomes

$$\left(\frac{2-b\cot\theta_j}{2b^2}\psi_{i,j+1} + \frac{2-a}{2a^2}\psi_{i+1,j} + \frac{2+b\cot\theta_j}{2b^2}\psi_{i,j-1} + \right. \tag{3.36}$$

$$\left.\frac{2+a}{2a^2}\psi_{i-1,j}\right) - \left(\frac{2}{a^2} + \frac{2}{b^2}\right)\psi_{i,j} = (e^{2z_i})(G_{i,j}),$$

and the finite-differenced constant properties vorticity-stream function equation becomes

$$e^{2z_i} \frac{G'_{i,j} - G_{i,j}}{\Delta t} + \left[\frac{\psi_{i+1,j} - \psi_{i-1,j}}{2a} \cdot \frac{F_{i,j+1} - F_{i,j-1}}{2b} - \right. \tag{3.37}$$

$$\left. \frac{F_{i+1,j} - F_{i-1,j}}{2a} \cdot \frac{\psi_{i,j+1} - \psi_{i,j-1}}{2b}\right] e^{z_i} \sin\theta_j$$

$$= \frac{2}{N_{Re}} \left[\left(\frac{2 - b \cot\theta_j}{2b^2} G_{i,j+1} + \frac{2-a}{2a^2} G_{i+1,j} + \frac{2 + b \cot\theta_j}{2b^2} G_{i,j-1} + \right.\right.$$

$$\left.\left. \frac{2+a}{2a^2} G_{i-1,j}\right) - \left(\frac{2}{a^2} + \frac{2}{b^2}\right) G_{i,j}\right],$$

where the prime denotes the value at the next time step.

Equation (3.37) is so executed as to establish the next step G field everywhere; the Equation (3.36) is executed iteratively using over-relaxation techniques in conjunction with the Gauss-Seidel approach to the solution of the Poisson's equation.

CHAPTER IV

DEVELOPING FLOW AROUND AN ISOTHERMAL SOLID SPHERE

4.1. Validity of the Solution

The solutions found using the numerical technique hitherto discussed exhibit excellent signs of convergence and stability. In addition to the absence of any signs of computational oscillation, the adopted computational technique shows a definite promise of tending towards a meaningful steady state solution. For example, Figure 10 illustrates the asymptotic character of maximum value of the stream function as the solution develops. Other factors such as the asymptotic development of the vorticity field as time proceeds further substantiate this conclusion. Stability, however, is conditional upon several factors. The least stringent of these appears to be the Courant number restriction, expressed by Roache (1972) for stretched coordinates as

$$\frac{1}{e^z} \frac{\partial \psi}{\partial z} \frac{\Delta t}{\Delta \theta} \leq 1. \tag{4.1}$$

This restriction is of minimal stringency and is only a necessary, not sufficient, condition for stability.

The second stability criterion relates the time step

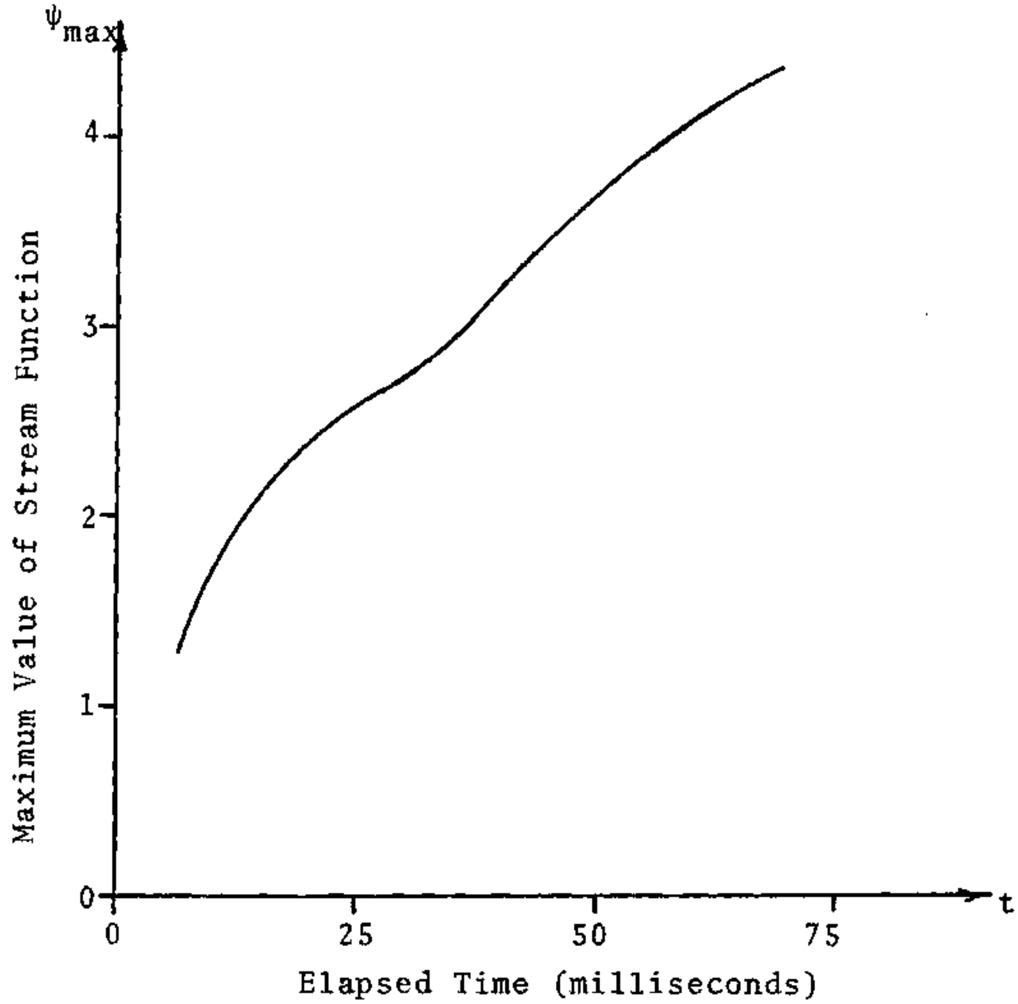


Figure 10. Peak Height of the Developing Stream Function

and the kinematic viscosity of the fluid system being modelled. The rate of diffusion of fluid momentum in laminar flow is represented by the kinematic viscosity of the fluid. The diffusion rate of a quantity may be related to the positional rate of change of its disturbance front. The disturbance front may be visualized as a spherical surface of arbitrary radius $r = r_o e^z$. The rate of change of the surface area, A, for such a sphere corresponding to the motion of the distrubance front may be expressed as

$$\frac{dA}{dt} = \frac{d}{dt}(4\pi r^2) = \frac{d}{dt}(4\pi r_o^2 e^{2z}) = 8\pi r_o^2 e^{2z} \frac{dz}{dt} . \qquad (4.2)$$

During the development of the numerical solution, it was observed that diffusion effects for the range of low Reynolds numbers extended to a distance generally less than five radii from the solid sphere surface. Since the vorticity diffusion rate must be compatible with the kinematic viscosity of the fluid system, the stability restriction for a radial step size, $\Delta z = a$, may be expressed as

$$\nu \leq 8\pi r_o^2 e^{2a} \frac{a}{\Delta t} . \qquad (4.3)$$

Equation (4.3) restricts the dimensional time step to 38.5 milliseconds for the case examined in this work ($a = 0.05$ and $N_{Re} = 15$, both dimensionless, $\nu = .255 \frac{cm^2}{sec}$ and $r_o = 0.084$ cm).

This requirement does not impose much stringency in its applicability, either.

The most stringent of the practical restrictions concerns the need for the vorticity function to behave in the expected diffusive manner at the solid surface. In order for this to occur during the initial time step when all field values are zero except on the boundary itself, Equation (3.37) is applied to the initial front of the vorticity disturbance. The vorticity function at the new time step, $G'_{2,j}$, must be related to its older surrounding vorticity function values. Because of the initial nature of the fields, the expression for the new vorticity function on the distrubance front is

$$G'_{2,j} = \frac{2c}{N_{Re}} \frac{1}{e^{2a}} G_{1,j}. \tag{4.4}$$

In order for $G'_{2,j}$ to be smaller in magnitude than $G_{1,j}$ for the case examined in this work, the dimensionless time must be less than 0.02022. By using only thirty per cent of such a value this requirement is well satisfied and supplies extremely reasonable time step values for stable solutions.

Hamielec, Johnson et al. (1967) have discussed in superb detail the spatial grid restrictions upon stability for steady flow problems in the transformed coordinate system used in this work. Computational grids of coarser mesh than those used in this work were found to cause oscillation of the vorticity function near the outer boundary downstream from the sphere. These oscillations were more predominant at larger Reynolds numbers. By using the appropriate spatial mesh sizes, no evidence of instability was observed. Furthermore, the recommended grid resolution of Hamielec, Johnson et al. proved quite satisfactory for the unsteady flow problem.

4.2. Consequences of the Solution

In order to interpret the solution, the time variant character of both the stream function and the vorticity function must be understood. Furthermore, the ability to correlate the observed phenomena with corresponding characteristics of the overall flow itself is of paramount importance. The discussion of the developing flow field pertains to the flow Reynolds number of fifteen (15). Precaution must be taken when viewing the three-dimensional plots illustrating the flow fields. These fields are plotted in a permuted cartesian grid, even though the values themselves represent quantities corresponding to the stretched spherical coordinate grid. A complete set of illustrations for the time-variant behavior of both the stream function and vorticity function is provided in Appendix E.

Preliminary Comments on the Solution

The steady state solution for the flow investigated in this work appears to require approximately 150 milliseconds to develop. This estimate of the characteristic time is based upon the analysis of the flow fields as they converged asymptotically. During this time several significant phenomena occur. Initially the stream function appears as a deformed elliptic paraboloid as it is visualized in Figure 11 as a surface in three-dimensional space. Within one-half of one percent of the characteristic time, the surface begins to lift itself upward, first in the forward region of the flow field, and ultimately in the rear region of the flow field. The result is to deposit a strong positive stream function field far from the sphere, and to maintain a narrow field of negative stream function immediately around the sphere itself.

As the flow continues to develop, the vorticity is entrained toward the rear of the flow field. This phenomenon begins at about ten percent of the characteristic time. By the end of fifty percent of completion time, the region of vorticity infected flow is developing into a parabolic region around the sphere with its focus at the sphere center. Inside the parabolic envelope the vorticity as well as the vorticity gradients are large, and hence viscous effects are significant. By this time the width of the vortex parabola is about five diameters at the shoulder of the sphere, demonstrating that the molecular diffusion of the vorticity possesses a length scale of comparable magnitude to the sphere body dimension. Figures 12, 13 and 14 illustrate the

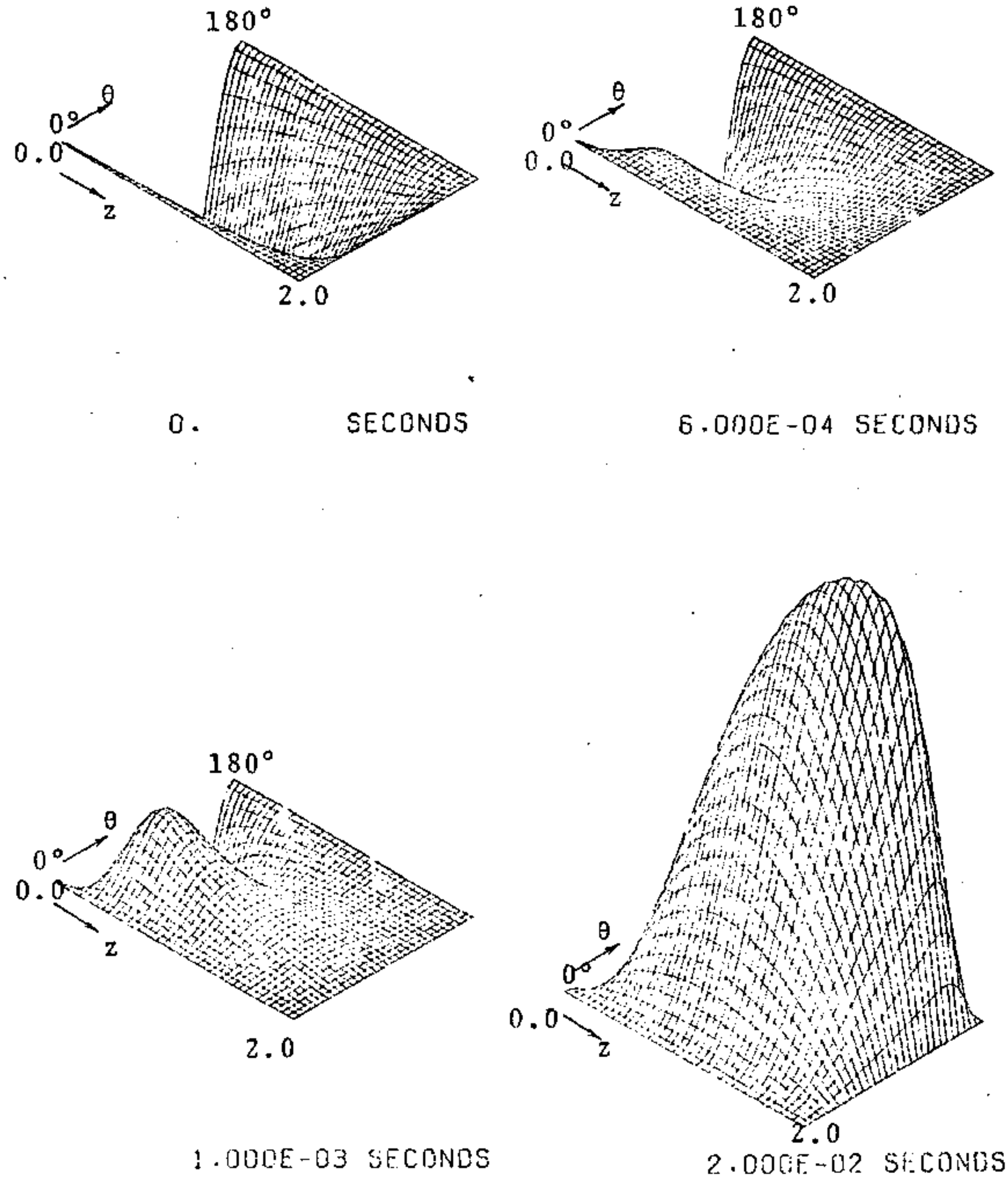


Figure 11. Developing External Stream Function as Three Dimensional Surface

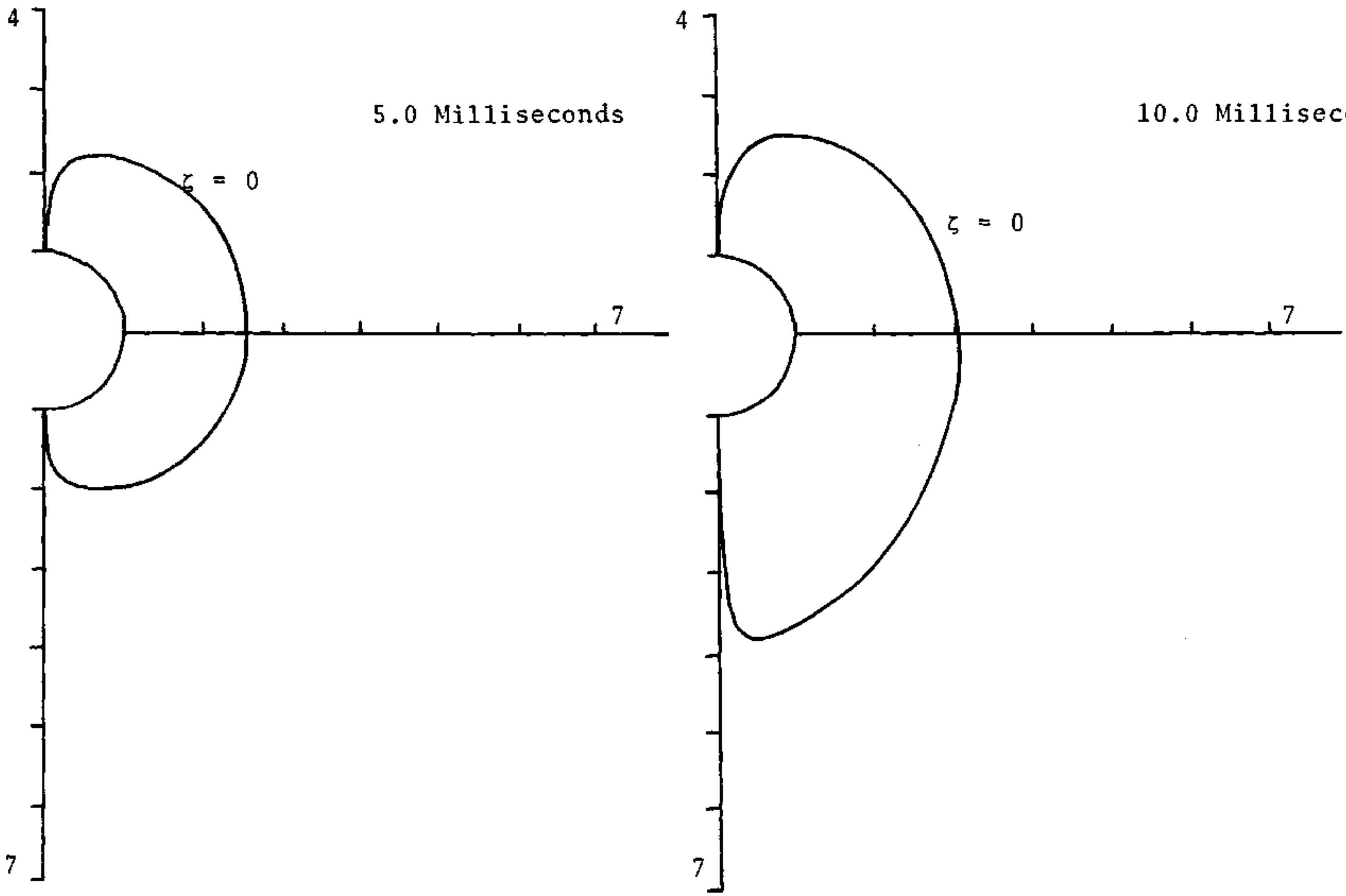


Figure 12. Developing Constant Vortex-lines (Re = 15)

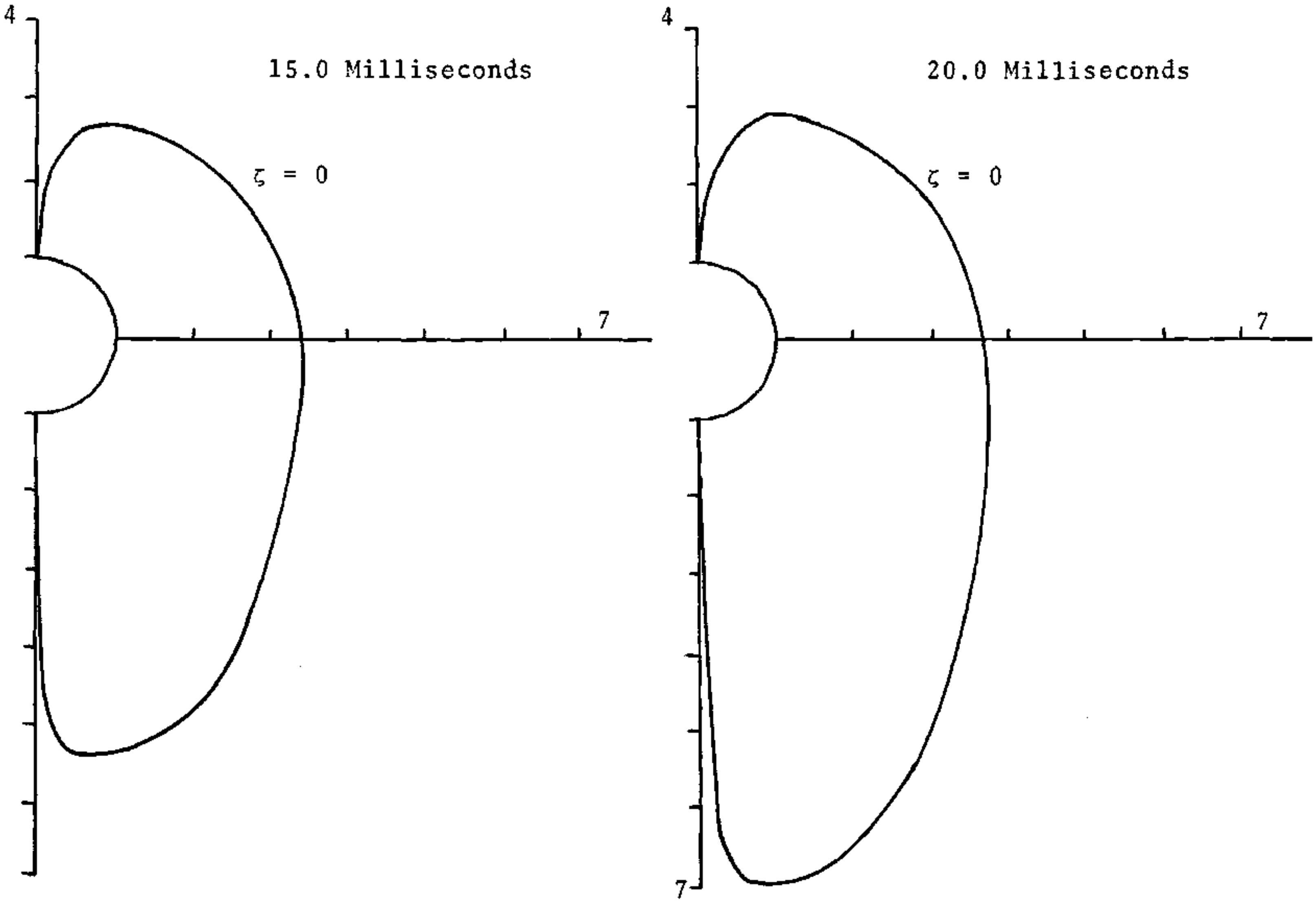


Figure 13. Developing Constant Vortex-lines (Re = 15)

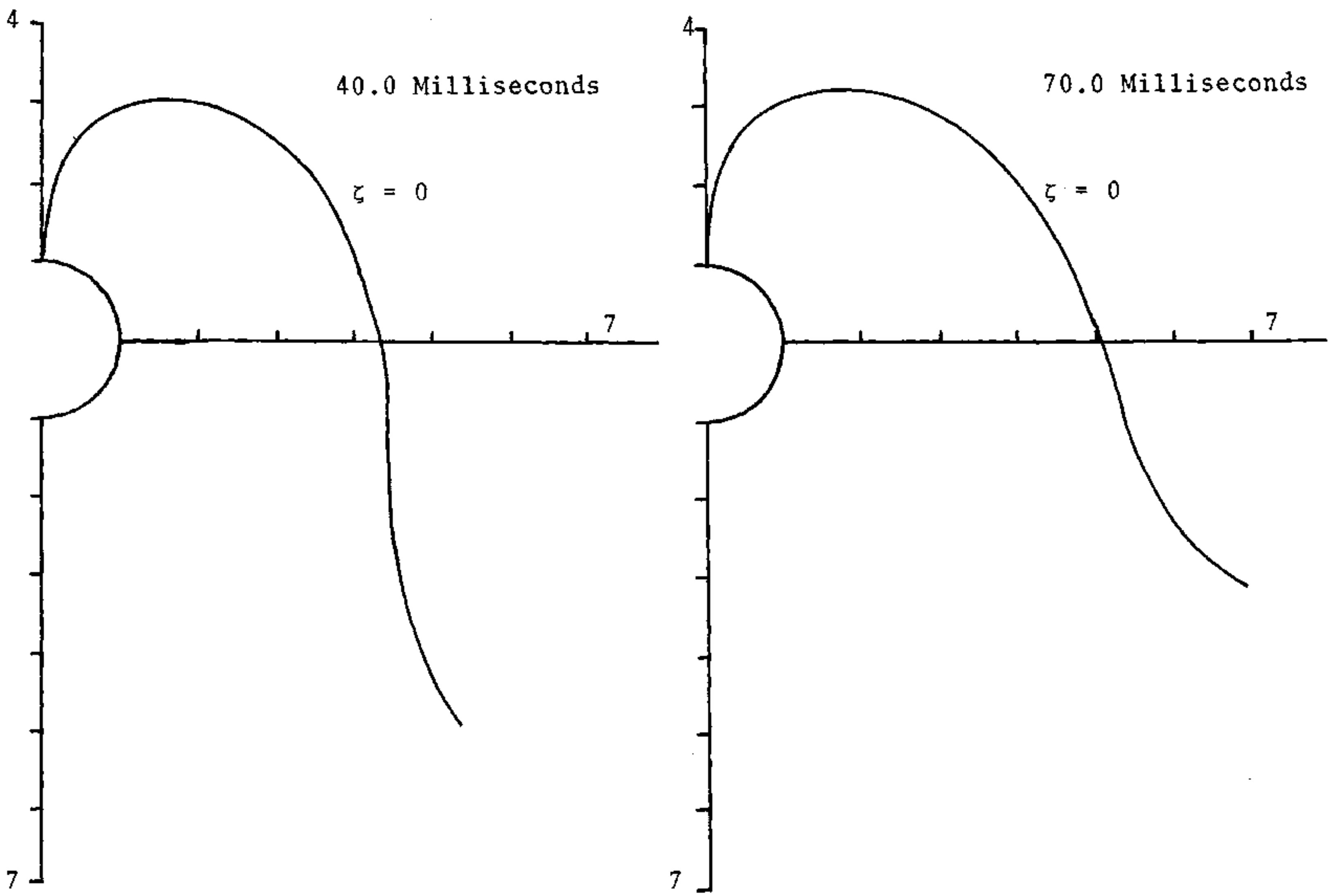


Figure 14. Developing Constant Vortex-lines (Re = 15)

progression in time of this region of vorticity infection. The vorticity is concentrated around the sphere and is contained within the parabolic envelope. Hence, the diffusion effects which are complemented by the presence of vorticity, are confined to the space immediately surrounding the sphere. Outside this envelope of vorticity, the flow behaves as an inviscid fluid. For this reason, Oseen's modification of the Stokes stream function for Reynolds numbers below five (5) were not totally accurate. In other words, it did not predict the wake region. The presence of this large region of concentrated vorticity was not compensated for in his model.

Furthermore, the vorticity envelope is being strongly deposited behind the sphere, which gives evidence that initiation of separation at the rear of the sphere is impending. Large gradients of vorticity cause the flow field to churn and swirl. When the vortex-lines themselves twist and stretch they introduce the cause for hydrodynamic instability which eventually leads to the transition to turbulent flow at large Reynolds numbers.

Finally, the flow examined in this work may be recognized as a problem of class I, described in Figure 2. By investigating a new class of problems the avenue of future research is therefore defined, not to mention the potential for fundamental research into the problems of classes II and III not examined in this work.

Analysis of the Stream Function

The initial stream function appears as a deformed elliptic paraboloid surface on a three dimensional grid shown in Figure 11. After about one-half of one percent of the characteristic time required for the solution to reach steady state, a region of positive value or prominence begins to emerge from the surface in the forward region of the flow field. This prominence continues to grow in magnitude toward an apparent asymptotic limit effectively reached after thirty percent of the characteristic time has elapsed. As this region of prominence emerges, its peak or maximum value is drifting in position and time. It originates at an angle of about 40 degrees and ultimately drifts in an almost parabolic path to an angle of about 140 degrees with respect to the forward stagnation point. Figure 15 illustrates this orderly movement of the peak as time proceeds.

Within one percent of the characteristic time, the initial streamlines themselves are swept toward the rear of the sphere. As Figures 16 and 17 illustrate, the streamlines very near the sphere surface are first affected, but very quickly the streamlines far removed from the sphere itself are entrained and deposited toward the rear of the flow field. After one percent of the characteristic time has passed, the region of prominence begins to dominate the flow field in both the forward and rear regions. By the end of five percent of the characteristic time, the negative

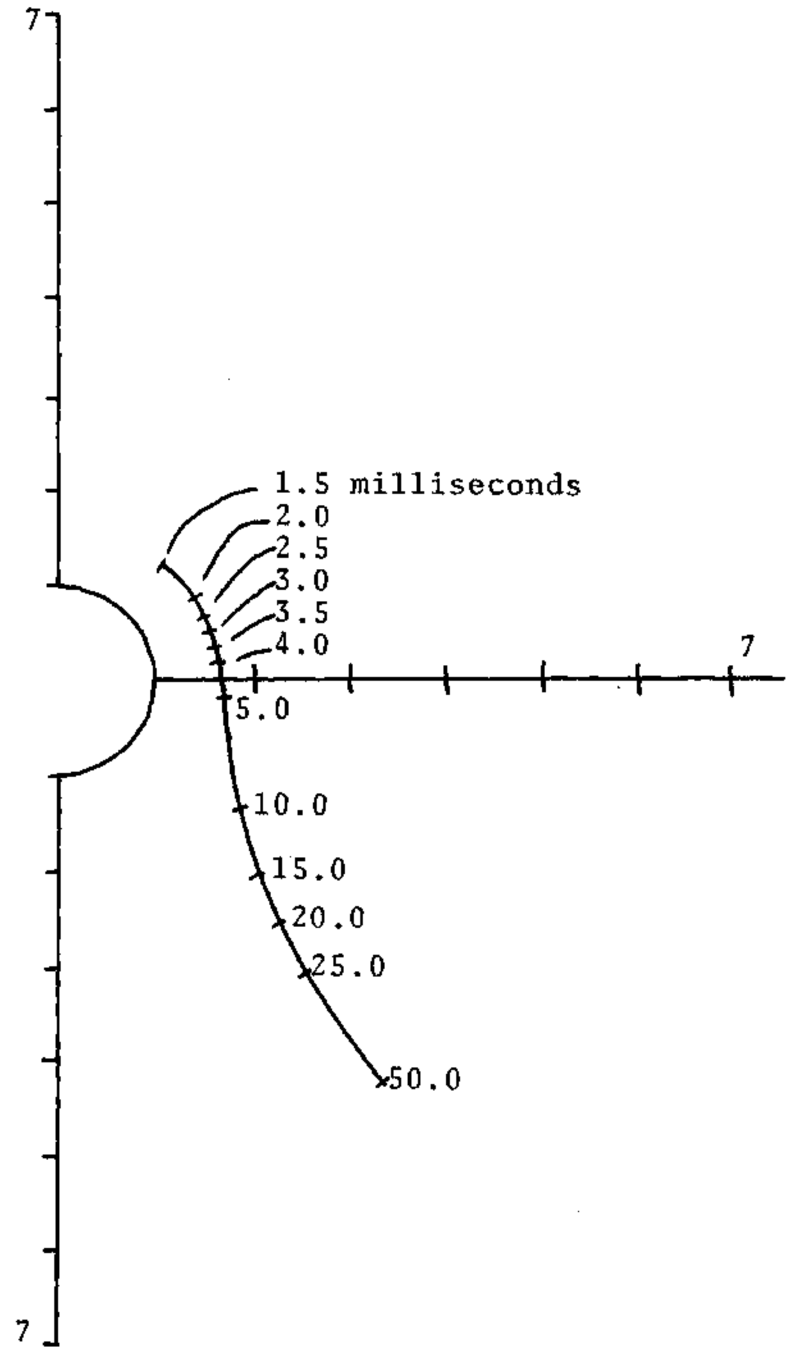


Figure 15. Path of the Stream Function Peak (Re = 15)

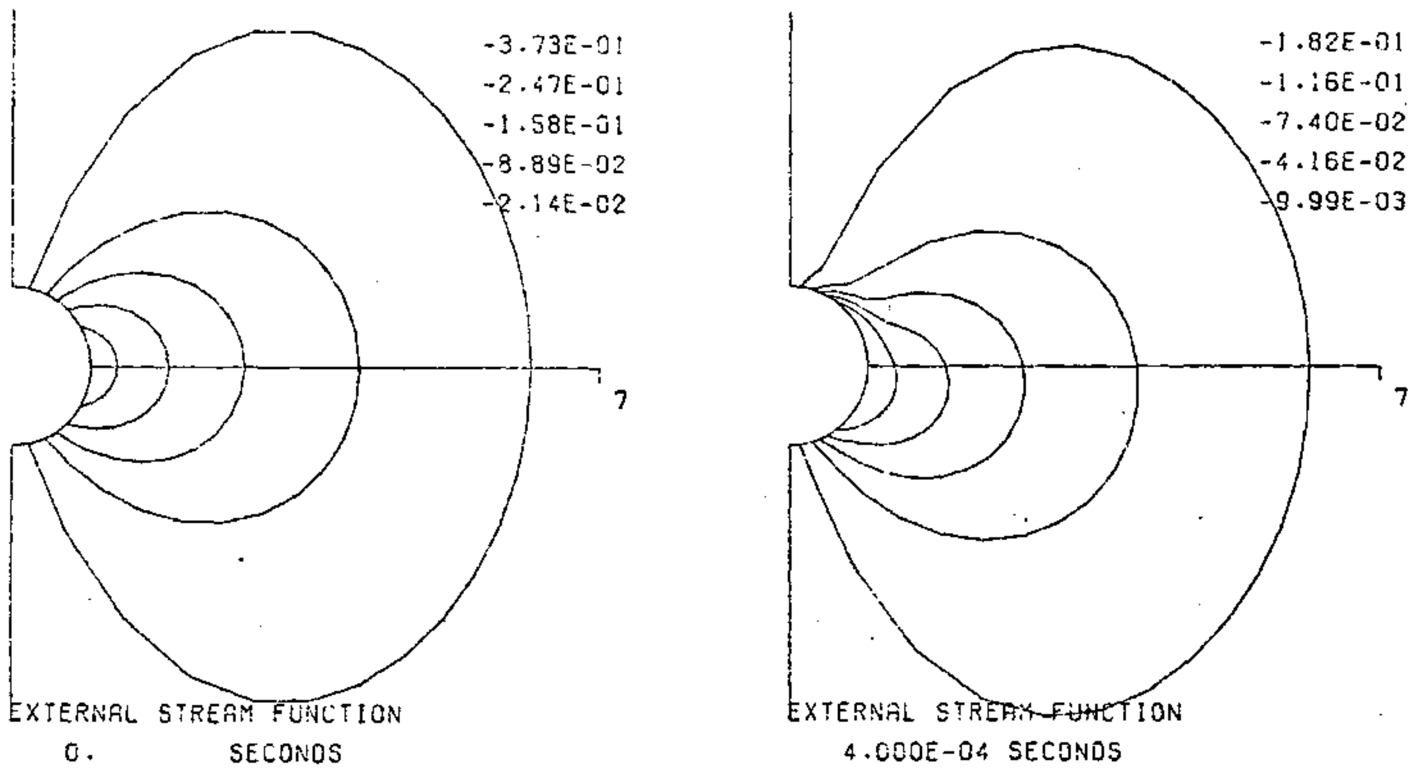


Figure 16. Developing External Stream Function as Lines of Constant Stream Function (Re = 15)

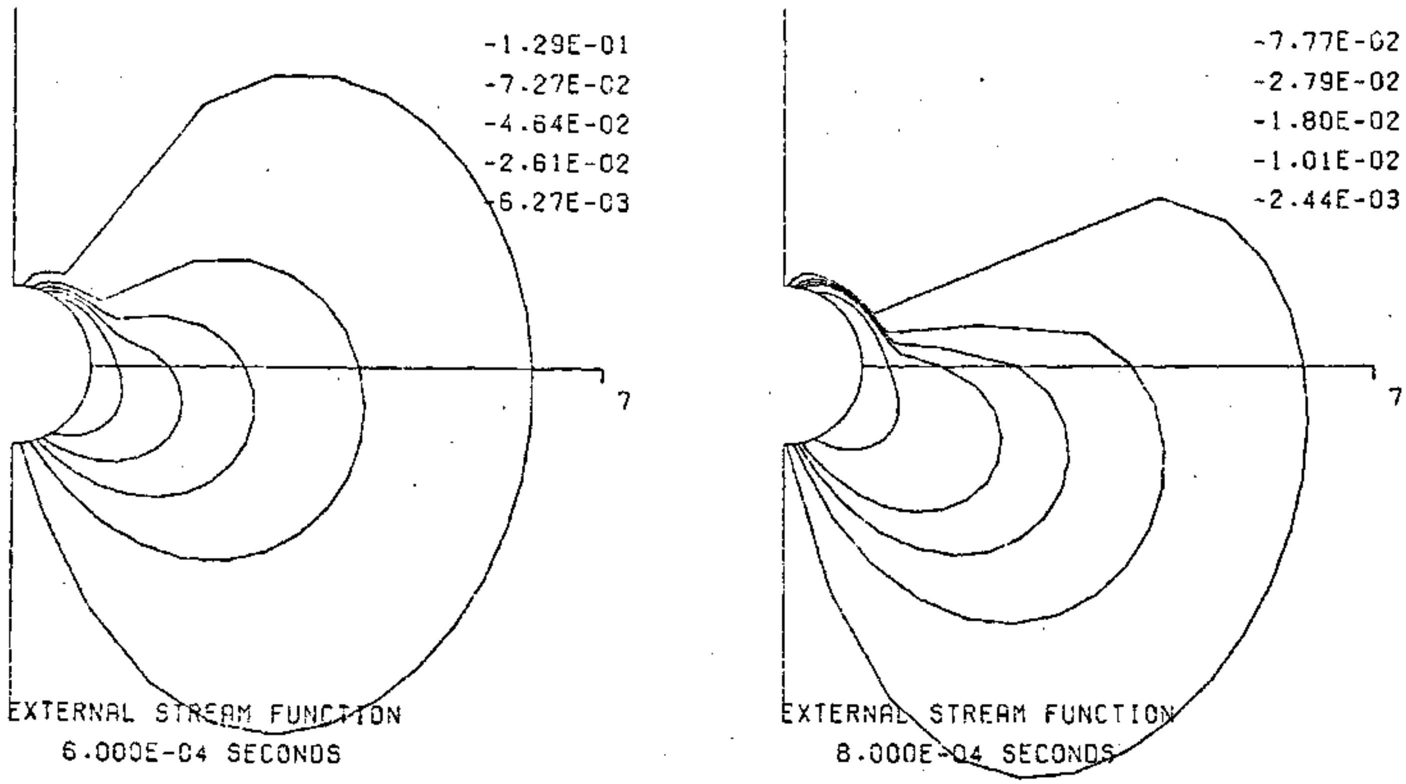


Figure 17. Developing External Stream Function as Lines of Constant Stream Function (Re = 15)

stream function values have been permanently confined to a narrow region very near to the sphere surface itself. Beyond this time the positive stream function values which fill most of the flow field far removed from the sphere characterize the flow field.

Analysis of the Vorticity Function

The vorticity appears initially as a bump or step distrubance along the sphere surface. As time proceeds, this disturbance propagates outward at a very moderate speed concentrating itself near the collar of the sphere. Figure 18 illustrates the characteristics of the developing external vorticity function as a three-dimensional surface. After ten percent of the characteristic time has elapsed, the vorticity function begins to be entrained toward the rear of the flow field. Figures 12, 13, 14 and 18 illustrate this phenomenon clearly. After about thirty percent of the characteristic time has elapsed, the vorticity has strongly concentrated itself in the region behind the sphere. At fifty percent of the characteristic time, this envelope of vorticity infection has taken on the familiar shape of a parabola with its focus at the center of the solid sphere. Batchelor (1967) discusses this phenomena from the steady-state viewpoint as a basic comparison between the characteristic length scales of diffusion and the suspended body itself. At low Reynolds numbers, the diffusion length scale is much larger than the body length scale. At large Reynolds numbers,

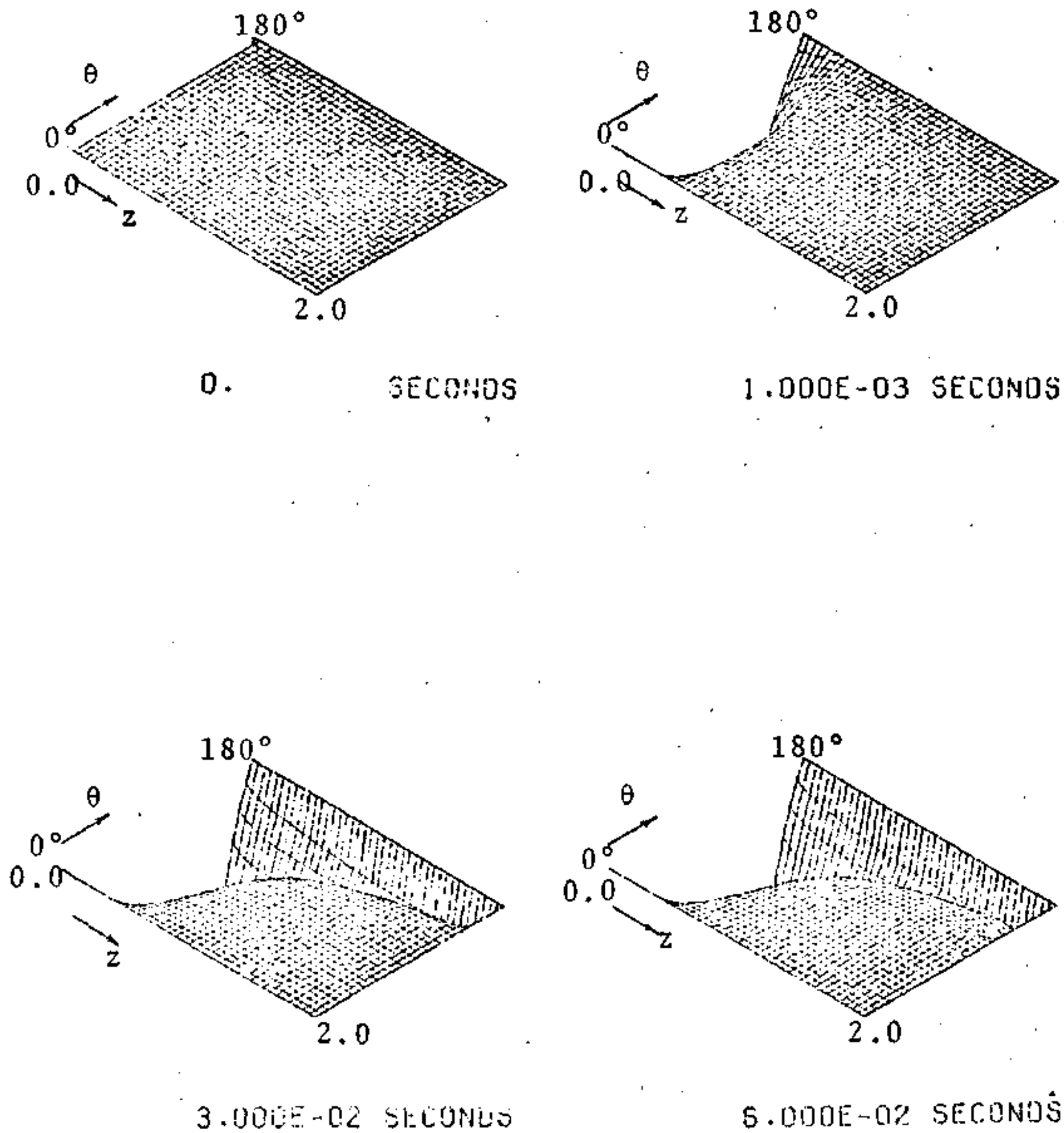


Figure 18. Developing External Vorticity Function as Three Dimensional Surface

(Re > 200), the body length scale is considerably larger than the diffusion length scale. For intermediate Reynolds number flow, (5 < Re < 200), the diffusion and body length scales are of the same order of magnitude, and hence diffusive and convective mechanisms are simultaneously influencing the flow field.

Up to about ten percent of the characteristic time the constant vortex-lines do not demonstrate any significant behavior. However, beyond this time the vortex-lines illustrated in Figures 12, 13 and 14 are being entrained toward the rear of the flow field. As the sphere proceeds forward, new vorticity is generated at the solid boundary. The old vorticity, though diffusing slowly, is still present, since the surrounding fluid cannot dissipate immediately the vorticity already created. Eventually the vorticity does diffuse and dissipate itself via frictional mechanisms, slowly eradicating the remaining vorticity left far behind the moving sphere.

4.3. Closure

As a closing remark on the nature of the solution obtained, it may be stated that dispite the symmetry of the boundary conditions imposed upon the solution, the solution ultimately develops asymmetric flow fields. Because of the presence of the convective terms in the governing differential equation, and because of the comparable diffusion and body length scales, both the vorticity function and the stream function develop their respective strong asymmetry. It may be further noted that the mathematical-numerical model is capable of displaying and does, in fact, display both the diffusive and convective nature of the very important regime of fluid mechanics, namely the flow for the range of Reynolds numbers between five (5) and 200.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

5.1. Conclusions

On the basis of this investigation it may be stated that the problem of the transient heat and momentum transport mechanisms between continuous and disperse media has been mathematically formulated. A solution technique has been formulated in broad terms to allow for its implementation to analyze many two-component direct-contact heat transfer problems. It has been further shown that the technique may be applied to solve the transient constant properties vorticity-stream function equation for the problem of the initial development of flow fields around an isothermal solid sphere instantaneously set into uniform motion at a Reynolds number less than 40 in an immiscible fluid medium. Furthermore, the initial and boundary conditions on a wide variety of conventional problems have been presented. These conditions have been presented in a form immediately applicable to the computational technique developed in this work.

The numerical solution of the developing flow around the sphere indicates that the stream function evolves into a region of positive value far away from the sphere, and a region of negative value very near the sphere surface. The vorticity, on the other hand, ultimately concentrates itself in a parabolic region behind the sphere. Despite the symmetry of the imposed boundary conditions, the convection of the flow causes the solution to eventually exhibit asymmetric stream and vorticity function fields.

The solution for the material rate of change of the vorticity of a fluid lump has demonstrated both the diffusion-like character and the convective character which is expected of viscous fluid motion for non-creeping flow problems. Since the solution exhibits no evidence of instability, it is certain that the further solution of the temperature equation and variable properties problem will proceed with comparable stability.

It is concluded that this work provides a definite background and direction for future research in the field of two-component direct-contact heat transfer. By achieving the level of sophistication detailed by the recommendations, the art of direct-contact heat transfer may ultimately become a practical science of great potential.

5.2. Recommendations

The potential usage and experimentation of the complete analytical-computational package developed in this work shows great promise. In order to achieve a high level of sophistication in the transport model, however, many independent subsystems must be evolved and coupled with the present system. As the development of the total transport model proceeds, its ability to accurately reflect more realistic and yet mathematically complicated transport phenomena should be assured.

Recommendations for Immediate Improvements

Several independent activities might be immediately initiated. A more efficient and concise technique to visually display the various variables involved in the transient solution need to be explored. Also, an implicit technique or a hopscotch technique to advance the time step solutions is likely to yield solutions at increased levels of computational efficiency as well as numerical stability. The actual implementation of careful and discrete numerical experimentation should be so developed as to help correlate these with laboratory experiments. In order to provide the model with realistic self-propagating characteristics. in-line evaluation and treatment of such gross flow parameters as pressure, stresses, and drag must ultimately be incorporated into the program.

Recommendations for Further Improvements

More complicated development activities are recommended toward ultimate growth of the program model. It may be noted that the adaption of the model to allow for a two-fluid system is complicated by the problems of the time-variant interfacial boundary conditions. A further inclusion of the temperature field into the model should allow for the solution of a great number of realistic problems of two-component direct-contact heat transfer. The variable viscosity effects may be more directly investigated after the temperature field is included in the analysis.

Recommendations for Long-Term Improvements

A considerably higher level of sophistication would entail the inclusion of the substance properties between phases into the model. This should allow for the modelling of heat transfer processes where one of the constituents changes phase from solid to liquid or liquid to gas during the process. Also, by modelling the transport properties of several bodies in motion in close proximity, the realistic problem of a two-component direct-contact heat transfer with phase change system such as the one proposed for geothermal energy production could be resolved.

Increased level of sophistication of the energy transport model can prove to be quite a powerful design tool. This may be achieved only through objective analysis and careful planning of the progress of the mechanism model.

APPENDICES

APPENDIX A

COMPONENTS OF THE STRESS TENSOR FOR NEWTONIAN FLUIDS IN SPHERICAL COORDINATES

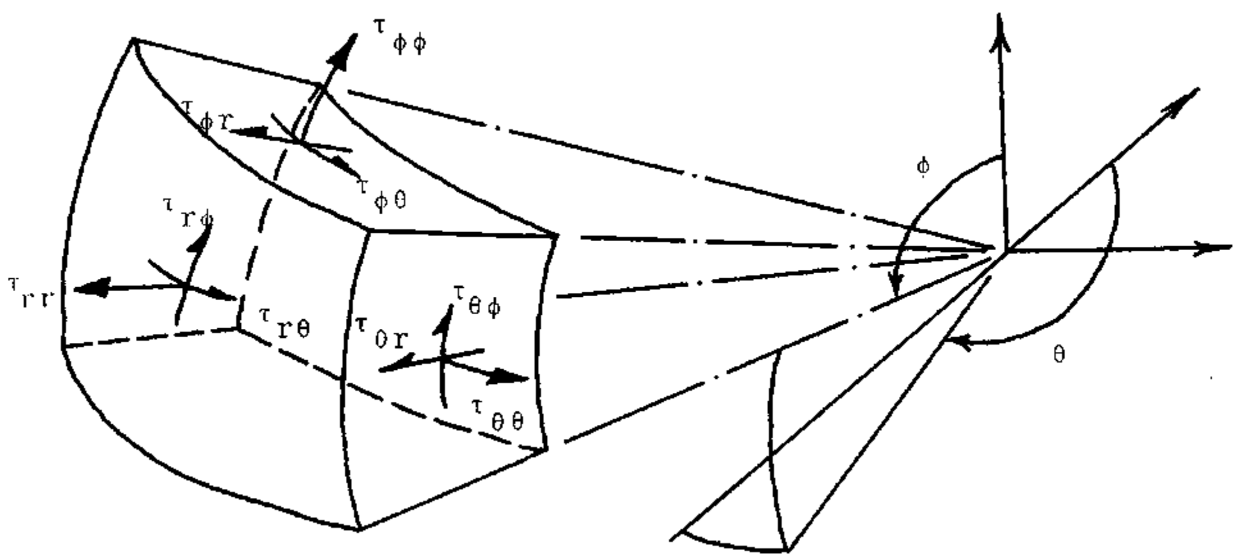


Figure 19. Stress Diagram of Fluid Element

The components of the stress tensor for Newtonian fluids in spherical coordinates, as shown in Figure 19, are

$$\tau_{rr} = -\mu[2 \frac{\partial v_r}{\partial r} - \frac{2}{3} (\nabla \cdot v)] \quad (A.1)$$

$$\tau_{\theta\theta} = -\mu[2(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}) - \frac{2}{3} (\nabla \cdot v)]$$

Equation (A.1) continued

$$\tau_{\phi\phi} = -\mu[2(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} + \frac{v_r}{r} + \frac{v_\theta\cot\theta}{r}) - \frac{2}{3}(\nabla\cdot v)] \qquad (A.1)$$

$$\tau_{r\theta} = \tau_{\theta r} = -\mu[r\frac{\partial}{\partial r}(\frac{v_\theta}{r}) + \frac{1}{r}\frac{\partial v_r}{\partial\theta}]$$

$$\tau_{r\phi} = \tau_{\phi r} = -\mu[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi} + r\frac{\partial}{\partial r}(\frac{v_\phi}{r})]$$

$$\tau_{\theta\phi} = \tau_{\phi\theta} = -\mu[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}(\frac{v_\phi}{\sin\theta}) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}].$$

Furthermore, for an incompressible fluid in axisymmetric flow with no ϕ-dependence, these components are

$$\tau_{rr} = -\mu[2\frac{\partial v_r}{\partial r}] \qquad (A.2)$$

$$\tau_{\theta\theta} = -\mu[2(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r}{r})]$$

$$\tau_{\phi\phi} = -\mu[2(\frac{v_r}{r} + \frac{v_\theta\cot\theta}{r})]$$

$$\tau_{r\theta} = \tau_{\theta r} = -\mu[r\frac{\partial}{\partial r}(\frac{v_\theta}{r}) + \frac{1}{r}\frac{\partial v_r}{\partial\theta}]$$

$$\tau_{r\phi} = \tau_{\phi r} = 0$$

$$\tau_{\theta\phi} = \tau_{\phi\theta} = 0.$$

APPENDIX B

COMPONENTS OF THE ENERGY FLUX VECTOR IN SPHERICAL COORDINATES

The components of the energy flux vector in spherical coordinates,

$$q_r = -k \frac{\partial T}{\partial r} \tag{B.1}$$

$$q_\theta = -k \frac{1}{r} \frac{\partial T}{\partial \theta}$$

$$q_\phi = -k \frac{1}{r \sin\theta} \frac{\partial T}{\partial \phi} ,$$

when applied to an incompressible fluid in axisymmetric flow with no ϕ-dependence, yields

$$q_r = -k \frac{\partial T}{\partial r} \tag{B.2}$$

$$q_\theta = -k \frac{1}{r} \frac{\partial T}{\partial \theta}$$

$$q_\phi = 0 .$$

APPENDIX C

DERIVATION OF THE VORTICITY RELATIONSHIP TO $E^2\psi$

In deriving the vorticity relationship to $E^2\psi$, the curl of the velocity is defined as the vorticity, thus the expression

$$(\vec{\nabla} X \vec{v})_\phi = 2(\vec{\omega})_\phi = 2\omega_\phi = +\zeta, \tag{C.1}$$

applies to the axisymmetric flow situation with no ϕ-dependence. After substitution,

$$(\vec{\nabla} X \vec{v})_\phi = \frac{1}{r} [\frac{\partial}{\partial r} (rv_\theta) - \frac{\partial}{\partial \theta} (v_r)], \tag{C.2}$$

may be transformed by the stream function relations (eq. 2.9), so that

$$2\omega_\phi = \frac{1}{r \sin\theta} [\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2\psi}{\partial\theta^2} - \frac{\cot\theta}{r^2} \frac{\partial\psi}{\partial\theta}] = \frac{E^2\psi}{r \sin\theta} = \zeta \tag{C.3}$$

may be expressed as

$$E^2\psi = \zeta \ r \sin\theta \ . \tag{C.4}$$

APPENDIX D

DERIVATION OF THE DIMENSIONAL VARIABLE PROPERTY STREAM FUNCTION EQUATION

By cross differentiation of the radial and tangential components of the momentum equation, eqs. (2.6) and (2.7) respectively, to eliminate the pressure terms, the resulting equation,

$$\frac{\partial}{\partial\theta}\left[\rho\left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta^2}{r}\right)\right] - \tag{D.1}$$

$$\frac{\partial}{\partial r}\left[\rho r\left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r v_\theta}{r}\right)\right]$$

$$= \frac{\partial}{\partial\theta}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\, 2\mu \frac{\partial v_r}{\partial r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\mu r \sin\theta \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \right.\right.$$

$$\left.\left.\frac{\mu\sin\theta}{r}\frac{\partial v_r}{\partial\theta}\right) - \frac{2\mu}{r^2}\left(2v_r + v_\theta\cot\theta + \frac{\partial v_\theta}{\partial\theta}\right)\right] - \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(\mu r^3 \frac{\partial}{\partial r}\right.\right.$$

$$\left(\frac{v_\theta}{r}\right) + \mu r \frac{\partial v_r}{\partial\theta}\Big) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(2\mu\sin\theta\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r}{r}\right)\right) + \mu r$$

$$\left.\left(\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r^2}\frac{\partial v_r}{\partial\theta}\right) - \frac{2\mu\cot\theta}{r}\left(v_r + v_\theta\cot\theta\right)\right],$$

may be manipulated into the more orderly form

$$\left(\frac{\partial v_r}{\partial t}\frac{\partial \rho}{\partial \theta} - r\frac{\partial v_\theta}{\partial t}\frac{\partial \rho}{\partial r}\right) + \rho\frac{\partial}{\partial t}\left(\frac{\partial v_r}{\partial \theta} - \frac{\partial}{\partial r}(rv_\theta)\right) \tag{D.2}$$

$$+ \left(v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r}\right)\frac{\partial \rho}{\partial \theta} - \left(rv_r\frac{\partial v_\theta}{\partial r} + v_\theta\frac{\partial v_\theta}{\partial \theta} + v_r v_\theta\right)\frac{\partial \rho}{\partial r}$$

$$+\rho\left[\frac{\partial}{\partial \theta}\left(v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r}\right) - \frac{\partial}{\partial r}\left(rv_r\frac{\partial v_\theta}{\partial r} + v_\theta\frac{\partial v_\theta}{\partial \theta} + v_r v_\theta\right)\right]$$

$$= \mu\left[\left(\frac{\partial^3 v_r}{\partial r^2\partial\theta} + \frac{1}{r^2}\frac{\partial^3 v_r}{\partial \theta^3} + \frac{\cot\theta}{r^2}\frac{\partial^2 v_r}{\partial\theta^2} - \frac{1+\cot^2\theta}{r^2}\frac{\partial v_r}{\partial\theta}\right)\right.$$

$$- \left(r\frac{\partial^3 v_\theta}{\partial r^3} + \frac{1}{r}\frac{\partial^3 v_\theta}{\partial r\partial\theta^2} + 3\frac{\partial^2 v_\theta}{\partial r^2} + \frac{\cot\theta}{r}\frac{\partial^2 v_\theta}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}\right.$$

$$\left.\left. - \frac{1+\cot^2\theta}{r}\frac{\partial v_\theta}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial v_\theta}{\partial\theta} - \frac{1+\cot^2\theta}{r^2}v_\theta\right)\right]$$

$$+ \left[\left(2\frac{\partial^2 v_r}{\partial r^2} + \frac{2}{r^2}\frac{\partial^2 v_r}{\partial\theta^2} + \frac{2}{r}\frac{\partial v_r}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{2}{r^2}v_r\right)\right.$$

$$\left. + \left(\frac{\cot\theta}{r}\frac{\partial v_\theta}{\partial r} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta} - \frac{3\cot\theta}{r^2}v_\theta\right)\right]\frac{\partial\mu}{\partial\theta}$$

$$+ \left[\left(-\frac{4}{r}\frac{\partial v_r}{\partial\theta}\right) + \left(-2r\frac{\partial^2 v_\theta}{\partial r^2} - \frac{2}{r}\frac{\partial^2 v_\theta}{\partial\theta^2} - 2\frac{\partial v_\theta}{\partial r} - \frac{2\cot\theta}{r}\frac{\partial v_\theta}{\partial\theta} + \right.\right.$$

$$\left.\left.\frac{2+2\cot^2\theta}{r}v_\theta\right)\right]\frac{\partial\mu}{\partial r} - \left[\left(\frac{\partial v_r}{\partial\theta}\right) + \left(r\frac{\partial v_\theta}{\partial r} - v_\theta\right)\right]\left(\frac{\partial^2\mu}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2\mu}{\partial\theta^2}\right)$$

$$+ \left[\left(2\frac{\partial v_r}{\partial r} - \frac{2}{r}v_r\right) - \left(\frac{2}{r}\frac{\partial v_\theta}{\partial\theta}\right)\right]\frac{\partial^2\mu}{\partial r\partial\theta} ,$$

which may be transformed, after substituting the stream function relations (2.9), into the dimensional variable property stream function equation,

$$\frac{1}{\sin\theta}\left(\frac{\partial^2\psi}{\partial t\partial r}\frac{\partial\rho}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial t\partial\theta}\frac{\partial\rho}{\partial\theta}\right)+\frac{\rho}{\sin\theta}\frac{\partial}{\partial t}(E^2\psi) \tag{D.3}$$

$$-\frac{1}{r\sin^2\theta}\left[\left(\frac{1}{r^2}\frac{\partial^2\psi}{\partial r\partial\theta}\frac{\partial\psi}{\partial\theta}-\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\frac{\partial\psi}{\partial r}-\frac{1}{r}\left(\frac{\partial\psi}{\partial r}\right)^2\right.\right.$$

$$\left.+\frac{\cot\theta}{r^2}\frac{\partial\psi}{\partial r}\frac{\partial\psi}{\partial\theta}-\frac{2}{r^3}\left(\frac{\partial\psi}{\partial\theta}\right)^2\right)\frac{1}{r}\frac{\partial\rho}{\partial\theta}$$

$$\left.+\left(\frac{1}{r}\frac{\partial^2\psi}{\partial r^2}\frac{\partial\psi}{\partial\theta}-\frac{1}{r}\frac{\partial^2\psi}{\partial r\partial\theta}\frac{\partial\psi}{\partial r}+\frac{\cot\theta}{r}\left(\frac{\partial\psi}{\partial r}\right)^2\right)\frac{\partial\rho}{\partial r}\right]$$

$$+\rho\,\frac{\partial\left(\psi,\dfrac{E^2\psi}{r^2\sin^2\theta}\right)}{\partial(r,\theta)}$$

$$=\frac{\mu}{\sin\theta}E^4\psi+\frac{1}{\sin\theta}\left[2\frac{\partial}{\partial r}\left(E^2\psi-\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right]\frac{\partial\mu}{\partial r}+\frac{1}{\sin\theta}\left[\frac{2}{r^2}\frac{\partial}{\partial\theta}(E^2\psi)\right.$$

$$\left.-\frac{\cot\theta}{r^2}(E^2\psi)-\frac{2}{r^2}(C^2\psi)\right]\frac{\partial\mu}{\partial\theta}+\frac{1}{\sin\theta}\left[2r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)-E^2\psi\right]$$

$$\left(\frac{\partial^2\mu}{\partial r^2}-\frac{1}{r^2}\frac{\partial^2\mu}{\partial\theta^2}\right)+\frac{1}{\sin\theta}\left[\frac{2}{r}(C^2\psi)\right]\frac{\partial^2\mu}{\partial r\partial\theta},$$

where,

$$E^4\psi = E^2(E^2\psi) = \frac{\partial^4\psi}{\partial r^4} + \frac{2}{r^2}\frac{\partial^4\psi}{\partial r^2\partial\theta^2} + \frac{1}{r^4}\frac{\partial^4\psi}{\partial\theta^4} - \frac{2\cot\theta}{r^2}\frac{\partial^3\psi}{\partial r^2\partial\theta} \tag{D.4}$$

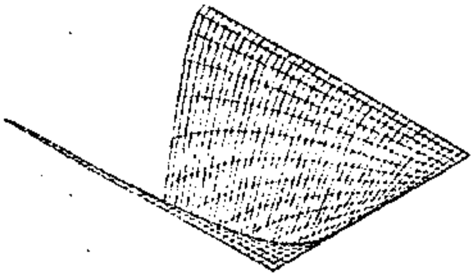
$$- \frac{4}{r^3}\frac{\partial^3\psi}{\partial r\partial\theta^2} - \frac{2\cot\theta}{r^4}\frac{\partial^3\psi}{\partial\theta^3} + \frac{4\cot\theta}{r^3}\frac{\partial^2\psi}{\partial r\partial\theta} + \frac{8+3\cot^2\theta}{r^4}\psi_{\theta\theta} - \cot\theta$$

$$\frac{9+4\cot^2\theta}{r^4}\frac{\partial\psi}{\partial\theta} \quad .$$

APPENDIX E

TIME-VARIANT BEHAVIOR OF STREAM FUNCTION AND VORTICITY FUNCTION

EXTERNAL STREAM FUNCTION



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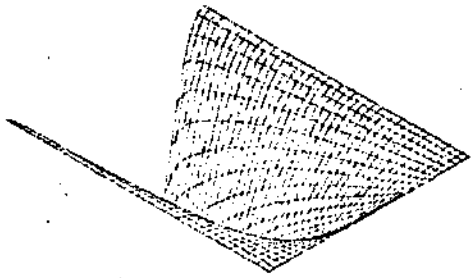

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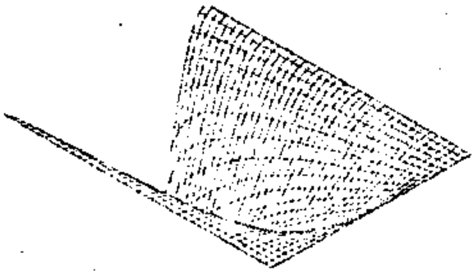

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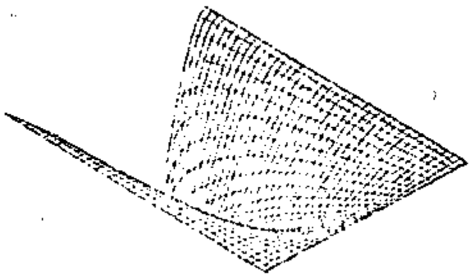

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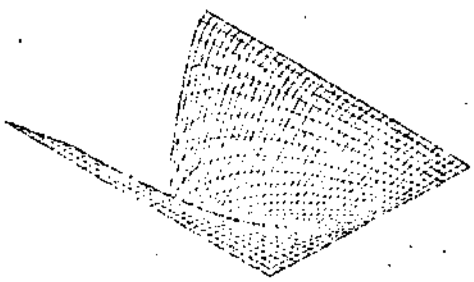

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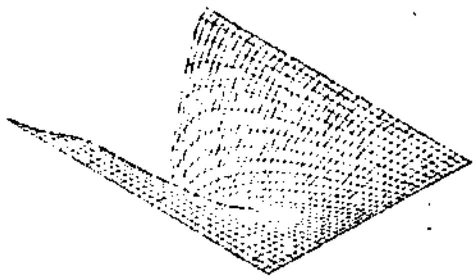

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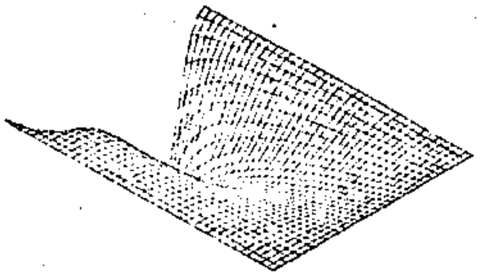
EXTERNAL STREAM FUNCTION

EXTERNAL STREAM FUNCTION.



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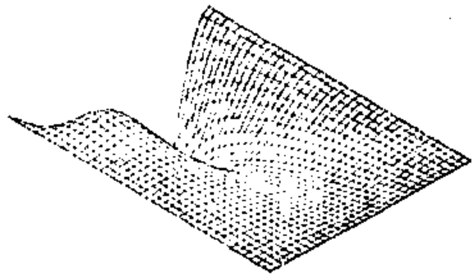

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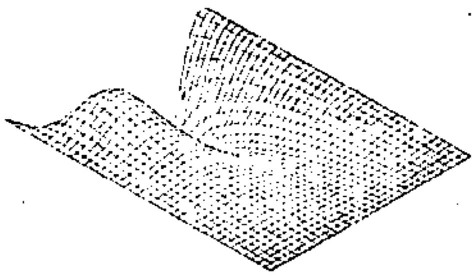

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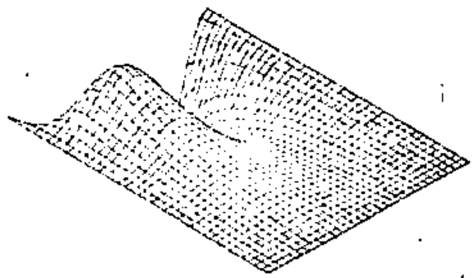

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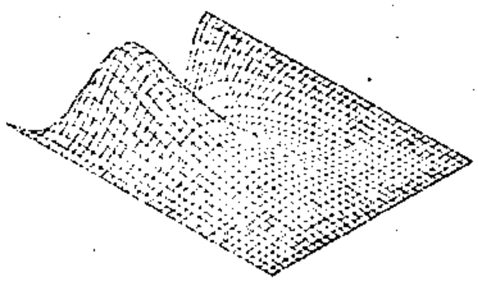

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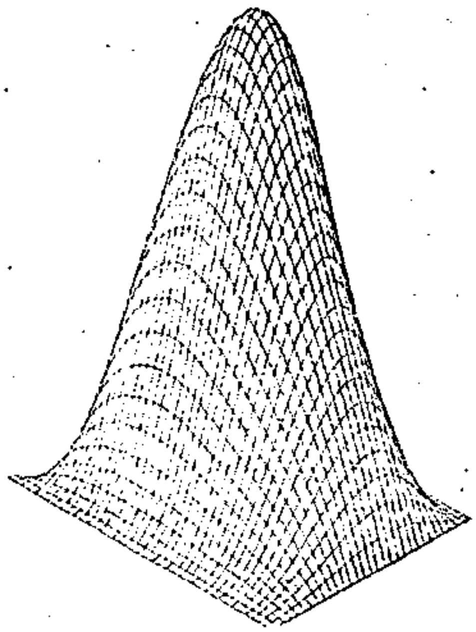
EXTERNAL STREAM FUNCTION

EXTERNAL STREAM FUNCTION



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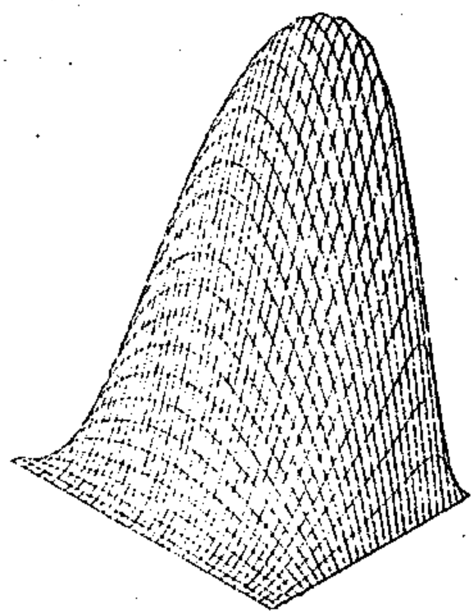

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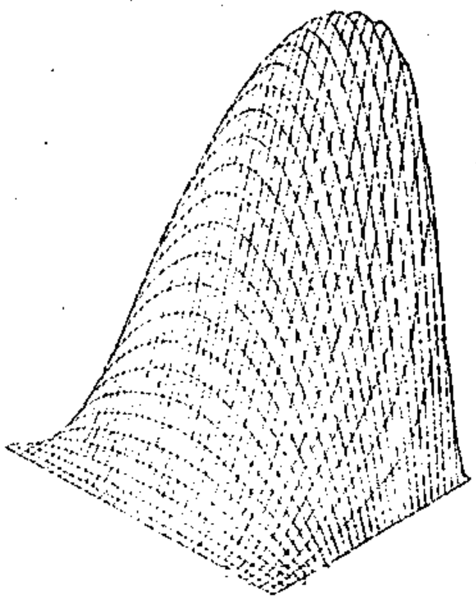

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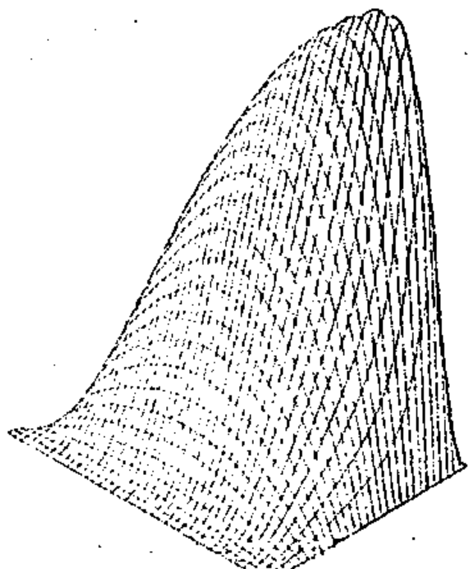

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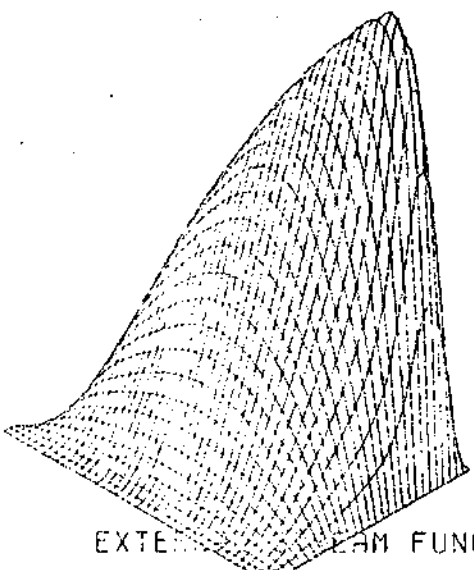
EXTERNAL STREAM FUNCTION

EXTERNAL STREAM FUNCTION



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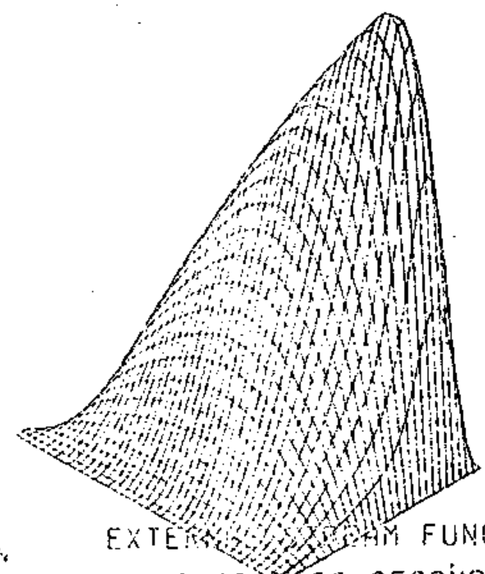

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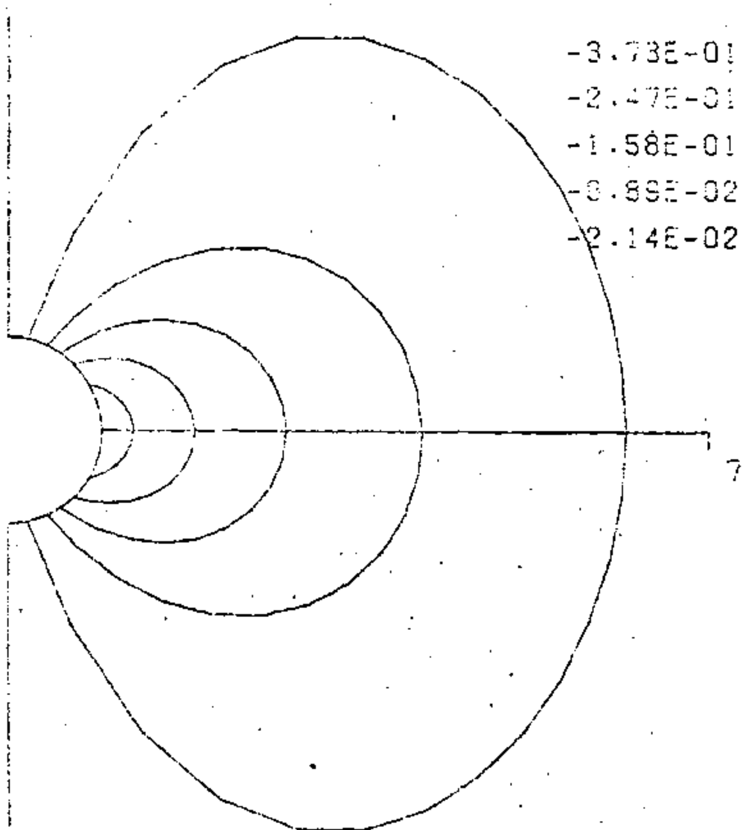

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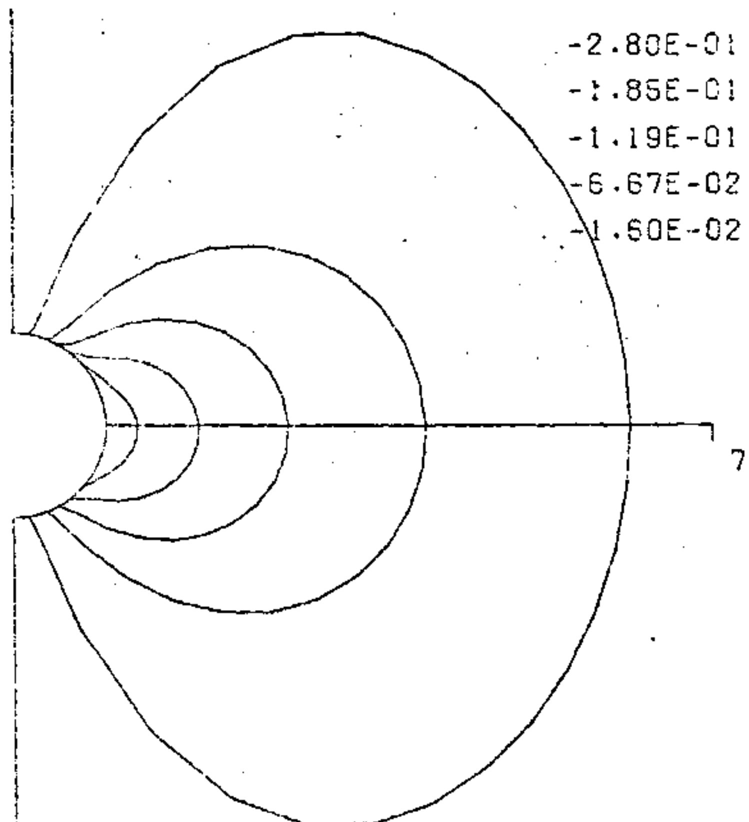

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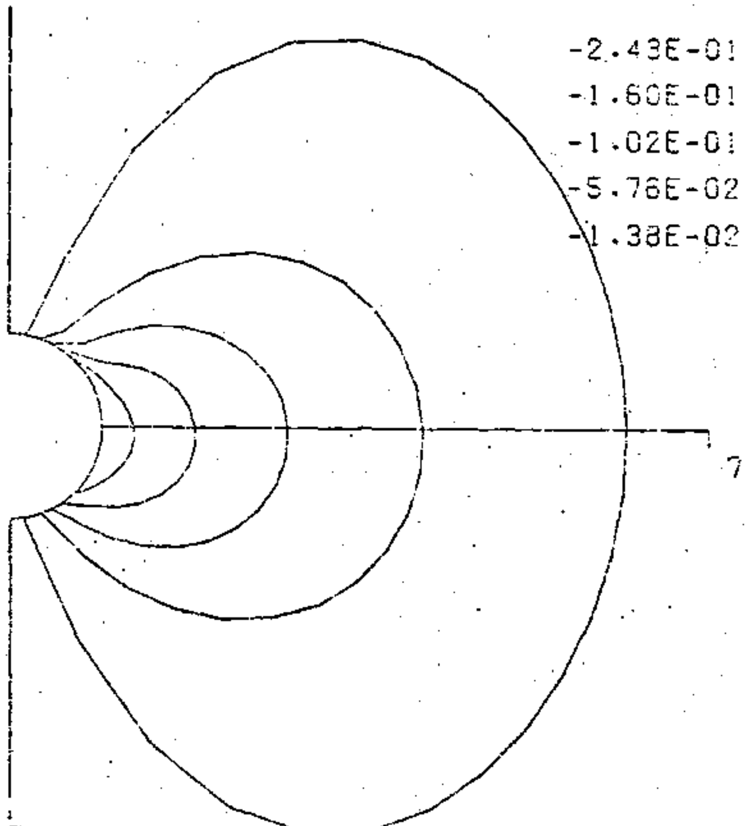
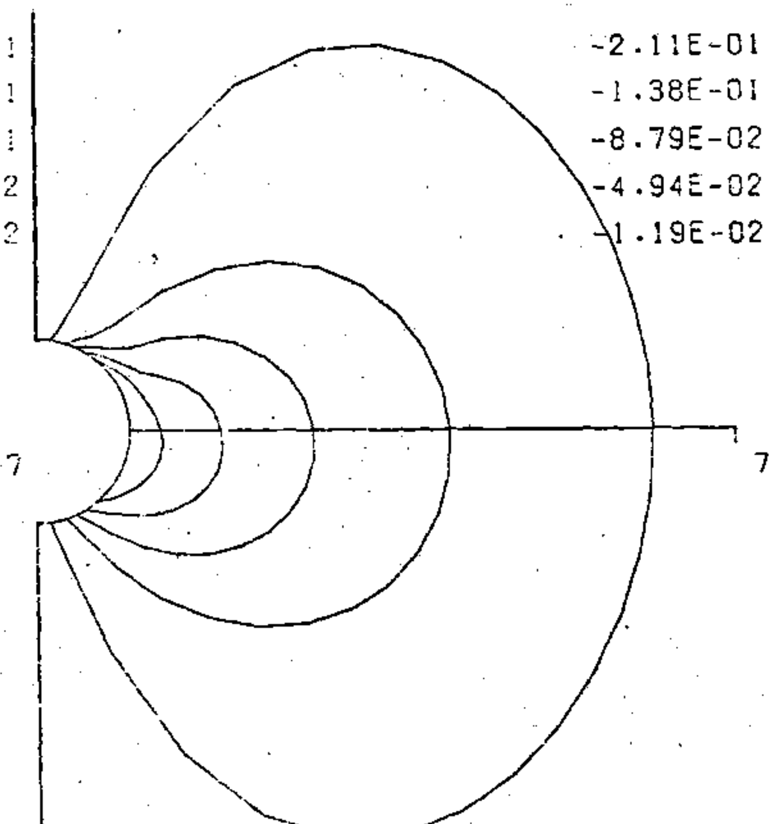


EXTERNAL STREAM FUNCTION
2.000E-04 SECONDS



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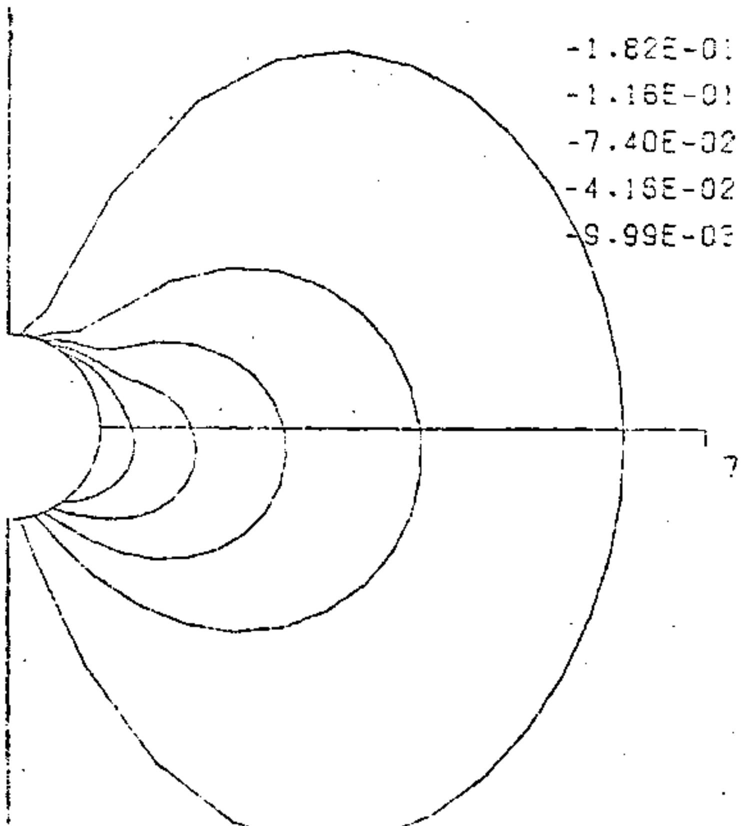
3.000E-04 SECONDS



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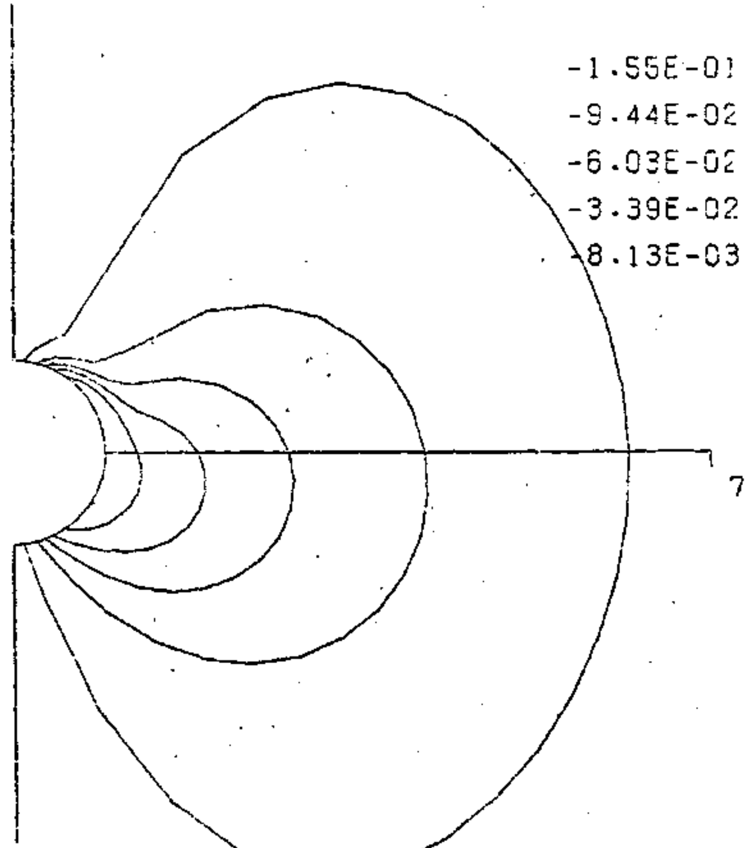

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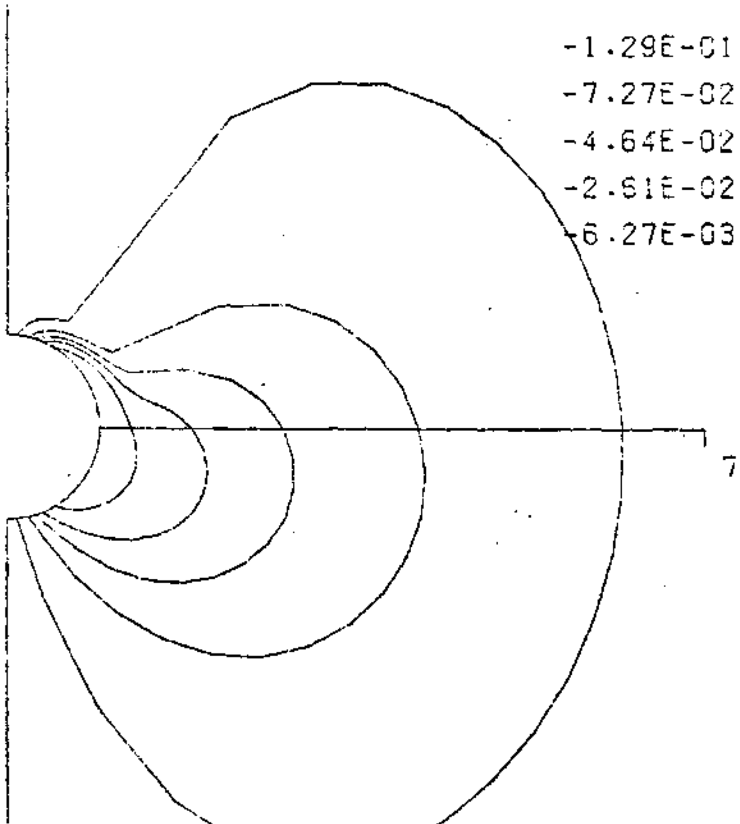
5.000E-04 SECONDS



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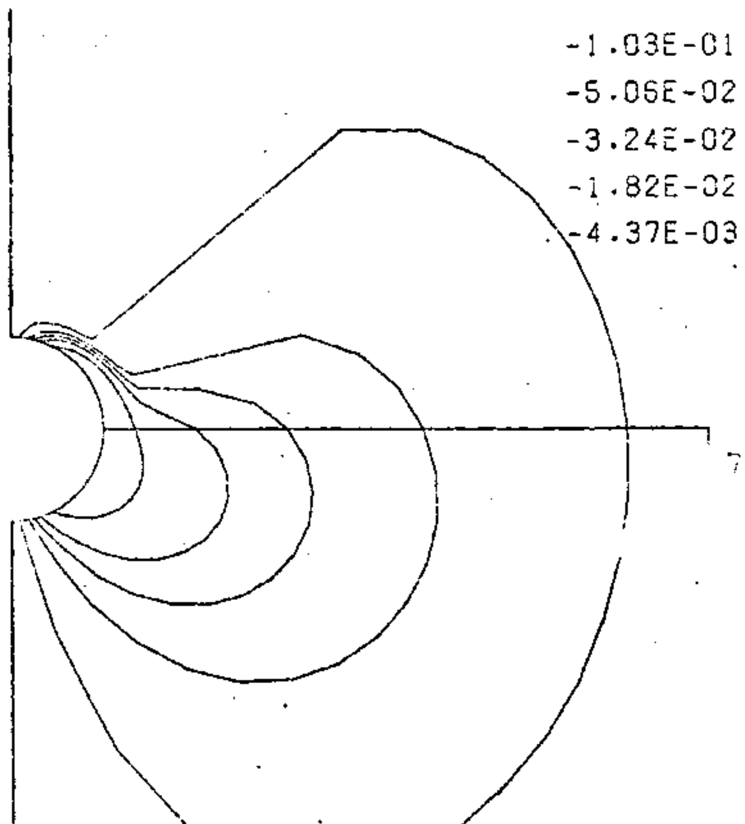

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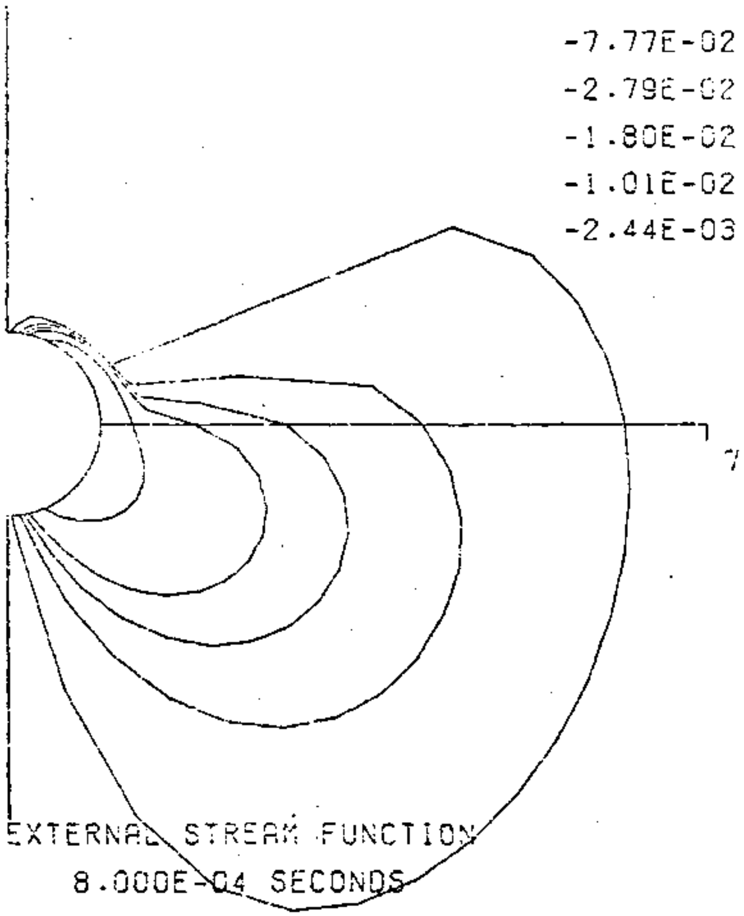

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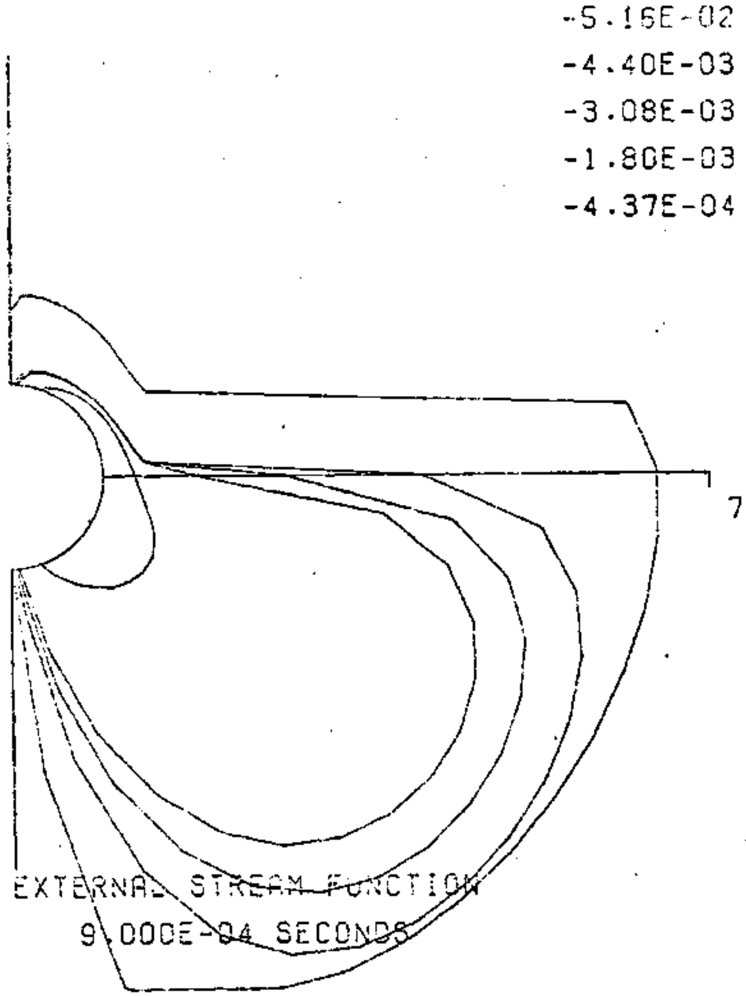
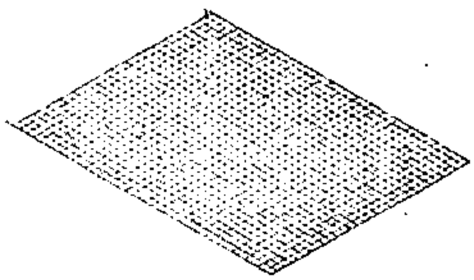


EXTERNAL STREAM FUNCTION
9.000E-04 SECONDS

EXTERNAL VORTICITY



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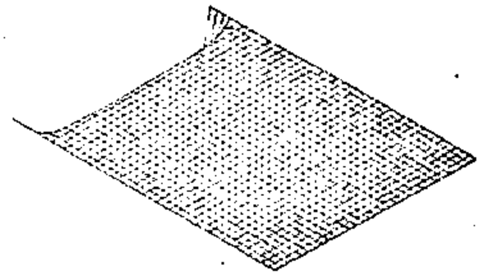

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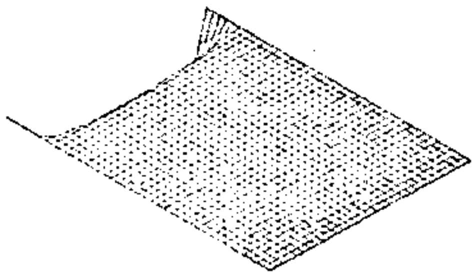

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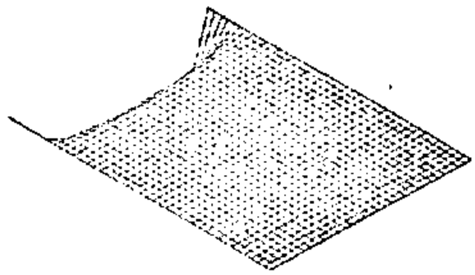

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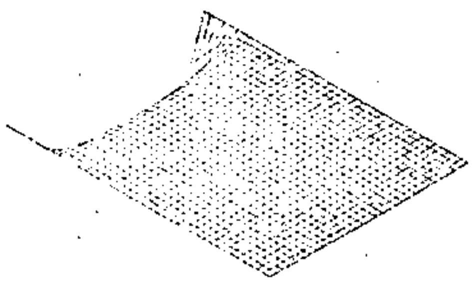

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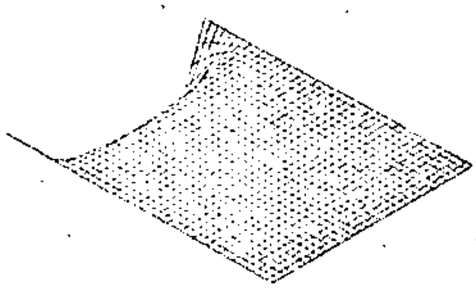
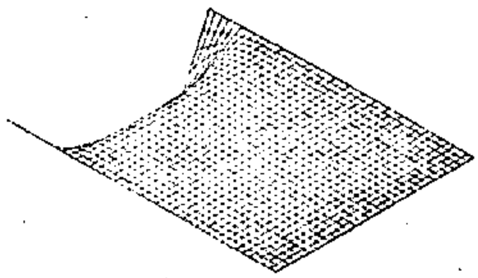


5.000E-04 SECONDS

EXTERNAL VORTICITY

EXTERNAL VORTICITY



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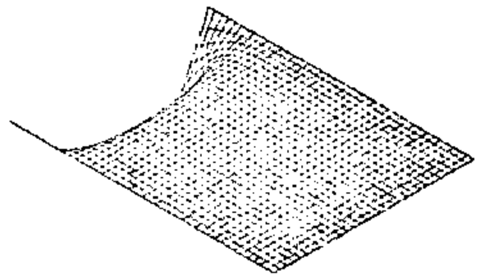

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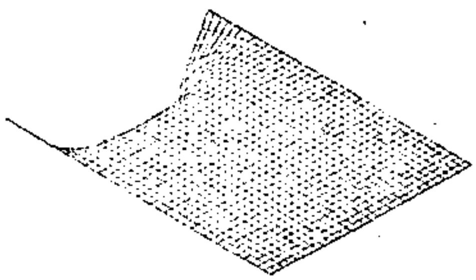

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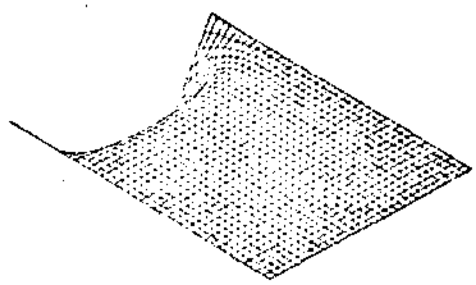

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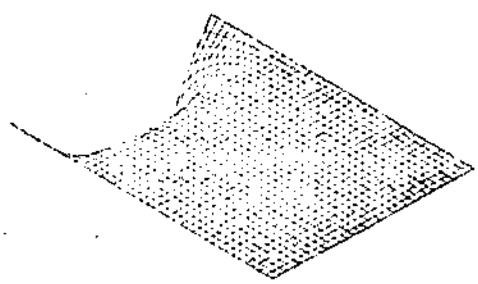
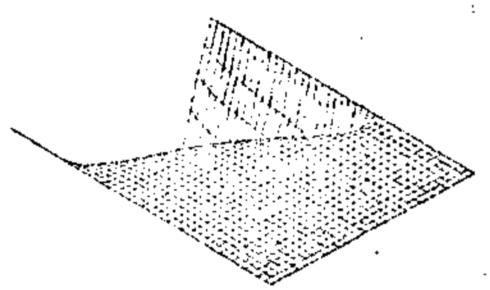


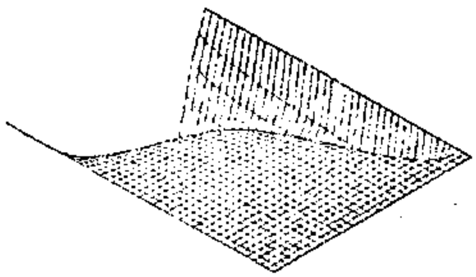
1.000E-03 SECONDS



1.000E-02 SECONDS

EXTERNAL VORTICITY

EXTERNAL VORTICITY



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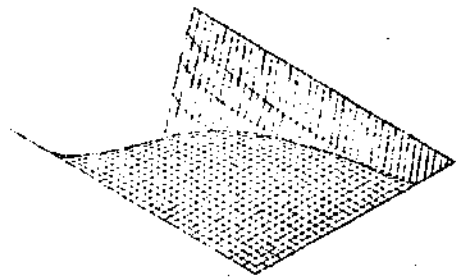

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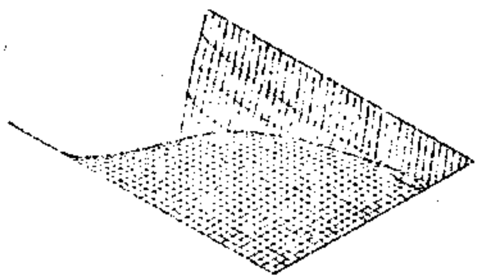

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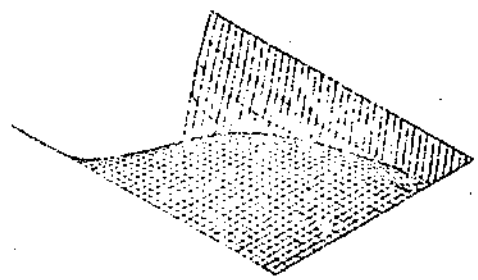
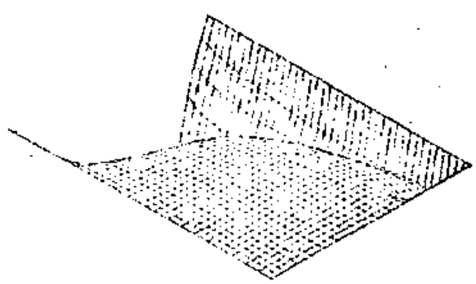


5.000E-02 SECONDS



6.000E-02 SECONDS

EXTERNAL VORTICITY

APPENDIX F

SELECTED FORTRAN SOURCE CODE PROGRAM LISTING

```
      SUBROUTINE POISSON ( PHI , PSI , ITMAX , EPS , TIME )
C23456789012345678901234567890123456789012345678901234567890123456789012
C        1         2         3         4         5         6         7
C.  THIS SUBROUTINE IS A MODIFICATION OF THE PROGRAM FOUND ON PAGES 491-
C. 497 OF "APPLIED NUMERICAL METHODS" BY CARNAHAN, LUTHER, & WILKES.  THE
C. PRINCIPAL DIFFERENCE IS NOT IN LOGIC BUT RATHER THIS SUBROUTINE SOLVES
C. THE POISSON SYSTEM IN SPHERICAL COORDINATES.  THE EQUATION SOLVED IS.
C.                ESQ(PHI)=E2(PHI)+PSI(Z,RAD).
C. IN A SPHERICAL RECTANGLE OF SIZE CHARACTERIZED AS:
C.      A  IN THE Z DIRECTION (ALONG WHICH THERE ARE M INCREMENTS).
C.      B  IN THE RAD DIRECTION (ALONG WHICH THERE ARE N INCREMENTS).
C. THE SPHERICAL OPERATOR ESQ OR E2 IS DEFINED ON PAGE 131 OF "TRANSPORT
C. PHENOMENA" BY BIRD, STEWART, & LIGHTFOOT (WITH THE EXCEPTION THAT WE
C. HAVE INTRODUCED A SPATIAL STRETCHING TRANSFORMATION TO ALLOW
C. R**EXP(Z) ).
C.  ITMAX GAUSS-SEIDEL ITERATIONS ARE PERFORMED AT ALL GRID POINTS.
C.
C.
C.    STOPS:    ***6***
C.
      COMMON /BLK/ M , N , MP1 , NP1 , MIDDLE , IWIGTH , A , B , C
      COMMON /FRAME/Z(41) , RAD(31) , EXPZ(41) , SINRAD(31) , COSRAD(31)
      COMMON AB , AA2 , AMI , APL , BB , BB2 , FMOVRN , MIDP1 , MM1 ,
     . NLINES , NM1 , OFFSET , PIOVRN , RELAX , RES , UOVR2A , UOVR2B
      DIMENSION PHI(41,31) , PSI(41,31)
C.
C.    EXECUTABLE STATEMENTS.
C.
         DO 3 ITER=1,ITMAX
         CNVRG = 0.0
            DO 2 I=2,M
            II=I
            EXPZI = EXPZ(I)
            FACT = EXPZI * EXPZI / RES
               DO 1 J=2,N
               JJ=J
               TEMP = QUAD(PHI,II,JJ) / RES - FACT * PSI(I,J)
               TEMP = PHI(I,J) + RELAX * ( TEMP - PHI(I,J) )
               CNVRG = CNVRG + ABS ( TEMP - PHI(I,J) )
               PHI(I,J) = TEMP
    1          CONTINUE
    2       CONTINUE
         IF ( CNVRG .LE. EPS ) GO TO 4
    3    CONTINUE
      WRITE(6,601) ITMAX , EPS , CNVRG , TIME
      CALL QUIT ( 6 )
C.TEMPWRITE(6,*)ITER
    4 RETURN
  601 FORMAT(1H1,I4,57H ITERATIONS OF POISSON HAS FAILED TO CONVERGE WIT
     .HIN EPS=,1PE10.3/33H CONVERGENCE HAS HALTED AT CNVRG=,1PE10.3,
     .15H FOR REAL TIME=,1PE14.4//13H ***STOP 6***)
      END
```

```
      SUBROUTINE WORKER ( W , F , G , ZETA , RE )
C23456789012345678901234567890123456789012345678901234567890123456789012
C        1         2         3         4         5         6         7
C.
C.    THIS SUBROUTINE COMPUTES THE NEW VORTICITY FIELD AT THE NEXT TIME
C.    STEP.  IT IS THE EMBODIMENT OF THE MAJOR GOVERNING PARTIAL
C.    DIFFERENTIAL EQUATION.
C.
      COMMON /BLK/ M , N , MP1 , NP1 , MIDDLE , IWIDTH , A , B , C
      COMMON /FRAME/Z(41) , RAD(31) , EXPZ(41) , SINRAD(31) , COSRAD(31)
      COMMON AA , AA2 , AMZ , APL , BB , BB2 , FPOVAN , MIDP1 , MM1 ,
     + NLINES , NM1 , OFFSET , PIOVAN , RELAX , RES , UOVRZA , UOVRZB
      DIMENSION W(41,31) , F(41,31) , G(41,31) , ZETA(41,31)
      DIMENSION GNU(41,31)
C.
C.    EXECUTABLE STATEMENTS.
C.
      CI = 2. * C / AE
         DO 2 II=2,M
         I = II
         EXPZI = EXPZ(I)
         CI = CI / EXPZI / EXPZI
            DO 1 JJ=2,N
            J = JJ
            GNU(I,J) = G(I,J)
     *                 + CI * ESQ( G , I , J )
     *                 + C * SINRAD(J) / EXPZI * JACOBI(F , W , I , J )
    1       CONTINUE
    2    CONTINUE
         DO 4 I=2,M
         EXPZI = EXPZ(I)
            DO 3 J=2,N
            C2 = EXPZI * SINRAD(J)
            G(I,J)=GNU(I,J)
            ZETA(I,J) = G(I,J) / C2
            F(I,J) = ZETA(I,J) / C2
    3       CONTINUE
    4    CONTINUE
      RETURN
      END
```

SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
3 WORKER

VARIABLES		SN	TYPE		RELOCATION				
6	A		REAL		BLK	0	AA	REAL	/ /
1	AA2		REAL		/ /	2	AMZ	REAL	/ /
3	APL		REAL		/ /	7	B	REAL	BLK
4	BB		REAL		/ /	5	BB2	REAL	/ /
10	C		REAL		BLK	313	CI	REAL	
280	COSRAD		REAL	ARRAY	FRAME	107	C1	REAL	

```
      FUNCTION ESQ ( W , I , J )
C2345678901234567890123456789012345678901234567890123456789012345678901
C        1         2         3         4         5         6         7
C.
C.    THIS FUNCTION COMPUTES THE FUNCTIONAL VALUE OF THE ALL IMPORTANT
C.    SPHERICAL SPATIAL DIFFERENTIAL OPERATOR ESQ.
C.
      COMMON AA , AA2 , AMI , APL , BB , BB2 , FHOVRN , MIOP1 , MM1 ,
     . NLINES , NM1 , OFFSET , PIOVRN , RELAX , RES , UOVR2A , UOVR2B
      DIMENSION W(41,31)
      ESQ = QUAD ( W , I , J ) - RES * W(I,J)
      RETURN
      END
```

```
SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
    4  ESQ

VARIABLES     SN  TYPE          RELOCATION
    0  AA         REAL              / /          1  AA2       REAL        / /
    2  AMI        REAL              / /          3  APL       REAL        / /
    4  BB         REAL              / /          5  BB2       REAL        / /
   27  ESQ        REAL                           6  FHOVRN    REAL        / /
    0  I          INTEGER           F.P.         0  J         INTEGER     F.P.
    7  MIOP1      INTEGER           / /         10  MM1       INTEGER     / /
   11  NLINES     INTEGER           / /         12  NM1       INTEGER     / /
   13  OFFSET     REAL              / /         14  PIOVRN    REAL        / /
   15  RELAX      REAL              / /         16  RES       REAL        / /
   17  UOVR2A     REAL              / /         20  UOVR2B    REAL        / /
    0  W          REAL       ARRAY  F.P.

EXTERNALS          TYPE   ARGS
      QUAD         REAL     3

COMMON BLOCKS   LENGTH
      / /          17

STATISTICS
  PROGRAM LENGTH                30B       24
  CM BLANK COMMON LENGTH        21B       17
```

```
      FUNCTION QUAD ( W , I , J )
C234567890123456789012345678901234567890123456789012345678901234567890112
C         1         2         3         4         5         6         7
C.
C.    THIS FUNCTION IS AN INTEGRAL AND SEPARABLE PART OF THE ALL
C.    IMPORTANT SPHERICAL SPATIAL DIFFERENTIAL OPERATOR EQ.
C.
      COMMON /BLK/ M , N , MP1 , NP1 , MIDDLE , INIGTH , A , B , C
      COMMON /FRAME/Z(41) , RAD(31) , EXPZ(41) , SINRAD(31) , COSRAD(31)
      COMMON AA , AA2 , AMI , APL , BB , BB2 , FROVAN , MIOP1 , MM1 ,
     . NLINES , NMI , OFFSET , PIOVRN , RELAX , RES , UOVR2A , UOVR2B
      DIMENSION W(41,31)
      BCOT = B * COSRAD(J) / SINRAD(J)
      QUAD = W(I,J+1) * (2. - BCOT ) / BB2
     .       +W(I+1,J) * AMI
     .       +W(I,J-1) * (2. + BCOT ) / BB2
     .       +W(I-1,J) * APL
      RETURN
      END
```

SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
4 QUAD

VARIABLES SN TYPE RELOCATION

SN	Name	Type		Relocation	SN	Name	Type		Relocation
6	A	REAL		BLK	0	AA	REAL		//
1	AA2	REAL		//	2	AMI	REAL		//
3	APL	REAL		//	7	B	REAL		BLK
4	BB	REAL		//	5	BB2	REAL		//
27	BCOT	REAL			10	C	REAL		BLK
220	COSRAD	REAL	ARRAY	FRAME	110	EXPZ	REAL	ARRAY	FRAME
6	FROVAN	REAL		//	3	I	INTEGER		F.P.
5	INIGTH	INTEGER		BLK	0	J	INTEGER		F.P.
0	M	INTEGER		BLK	4	MIDDLE	INTEGER		BLK
7	MIOP1	INTEGER		//	10	MM1	INTEGER		//
2	MP1	INTEGER		BLK	1	N	INTEGER		BLK
11	NLINES	INTEGER		//	12	NMI	INTEGER		//
3	NP1	INTEGER		BLK	13	OFFSET	REAL		//
14	PIOVRN	REAL		//	26	QUAD	REAL		
51	RAD	REAL	ARRAY	FRAME	15	RELAX	REAL		//
16	RES	REAL		//	161	SINRAD	REAL	ARRAY	FRAME
17	UOVR2A	REAL		//	20	UOVR2B	REAL		//
0	W	REAL	ARRAY	F.P.	0	Z	REAL	ARRAY	FRAME

COMMON BLOCKS LENGTH

Block	Length
BLK	9
FRAME	175
//	17

STATISTICS

PROGRAM LENGTH	30B	24
CM LABELLED COMMON LENGTH	270B	184

```
FUNCTION JACCBI     74/74   OPT=2                       FTN 4.6+424       77/03/16. 23.09.35     PAGE   1

          FUNCTION JACCBI ( F , W , I , J )
       C23456789012345678901234567890123456789012345678901234567890123456789012
       C        1         2         3         4         5         6         7
       C.
       C.   THIS SUBROUTINE COMPUTES THE JACOBIAN PARTIAL DERIVATIVE OPERATION
       C.     D(F,W) / D(Z,THETA).
       C.
          COMMON /BLK/ M , N , MP1 , NP1 , MIDDLE , IWIDTH , A , B , C
          DIMENSION F(41,31) , W(41,31)
          JACCBI = ( (F(I+1,J) - F(I-1,J)) * (W(I,J+1) - W(I,J-1))
         .          -(F(I,J+1) - F(I,J-1)) * (W(I+1,J) - W(I-1,J)) ) /4./A/B
          RETURN
          END


     SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
    4  JACOBI

VARIABLES      SN  TYPE            RELOCATION
    6  A           REAL                BLK          7  B        REAL               BLK
   10  C           REAL                BLK          0  F        REAL      ARRAY    F.P.
    0  I           INTEGER             F.P.         5  IWIDTH   INTEGER            BLK
    0  J           INTEGER             F.P.        26  JACOBI   INTEGER
    0  M           INTEGER             BLK          4  MIDDLE   INTEGER            BLK
    2  MP1         INTEGER             BLK          1  N        INTEGER            BLK
    3  NP1         INTEGER             BLK          0  W        REAL      ARRAY    F.P.

COMMON BLOCKS   LENGTH
      BLK            9

STATISTICS
  PROGRAM LENGTH                 27B      23
  CM LABELED COMMON LENGTH       11B       9
```

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